

Introduction to
GENERAL TOPOLOGY

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PREFACE

Topology is now included in the undergraduate curricula of most universities. The present book, just like our *Elements of General Topology* (which will be denoted by [H2]), is designed as a text for a one-semester or two-quarter course in general topology for upper division undergraduates as well as first-year graduate students. Its aim is to provide a systematic exposition of the essentials of this subject in a desirably leisurely fashion, to students who have reached at least the level of mathematical maturity following two or three years of sound undergraduate mathematics study.

After the publication of [H2], several professors teaching general topology suggested an edition of the book which would be suitable for the training of research mathematicians who would specialize in analysis as well as in algebra or topology. The result is this rather standard version of the earlier book.

The first three chapters of this book cover the more or less standard material on topological spaces and are exactly the same as the first three chapters of [H2] except that metric spaces and metrizability are now studied in Chapter IV. Instead of covering as much ground as possible, we elect to put almost all emphasis on the absolutely basic concepts, fundamental properties, and important constructions, presenting them carefully and painstakingly. In giving proofs, we include details which might ordinarily be omitted, preferring intelligibility over brevity. The examples are chosen so as to be used later and to produce minimal distraction from the main lines of development. Readers who are interested in pathological examples can find plenty of these in the existing books on the subject.

The fourth chapter offers a rather elaborate coverage of metrizability and metric spaces, including completeness and completions. To generalize the metric properties which are lost in the process of extending the notion of metric spaces to that of topological spaces, we present in Chapter V the axiomatic approach to uniformity and boundedness. The former is now standard while the latter has here its first appearance in book form. As illustrative examples of topological spaces with both a

natural uniformity and a natural boundedness, we study in the final chapter topological linear spaces and, as the important special case, normed linear spaces. These are treated in the same style as the topics in the first three chapters.

Including the exercises at the end of each section, enough material is presented here for a standard two-semester course. Together with the last three chapters of [H2], one can offer a very substantial year course of general topology to well-prepared students.

The bibliography at the end of the book comprises two parts, "Books" and "papers." The former contains almost all books on general topology, while the latter has been reduced to the minimum essential to the text and the exercises. References to the bibliography are letters and numbers enclosed in brackets. Cross references are given in the form (III, 8.1), where III stands for Chapter III and 8.1 for the numbering of the statement in the chapter. The chapter numeral will be omitted in case the reference is made in the same chapter.

A list of special symbols and abbreviations used in this book is given immediately after the Table of Contents. Certain deviations from standard set-theoretic notations have been adopted in the text; namely, $\square$ is used to denote the empty set and $A \setminus B$ the set-theoretic difference usually denoted by $A - B$. We have used the symbol $\parallel$ to indicate the end of a proof and the abbreviation *iff* for the phrase "if and only if."

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SPECIAL SYMBOLS AND ABBREVIATIONS

$\Rightarrow$	implies	
$\Leftarrow$	is implied by	
$\parallel$	end of proof	
iff	if and only if	
$\in$	is a member of	2
$\notin$	is not a member of	2
$\subset$	is contained in	2
$\supset$	contains	2
$=$	is equal to	3
$\cup$	union	4
$\cap$	intersection	4
Π	Cartesian product	12
$\{\}$	set	2
$\{\}$	set such that	2
$\square$	empty set	2
$\sim$	is equivalent to	89
$\simeq$	is homotopic to	48
$\cong$	is isotopic to	48
$\parallel$	norm of	204
2^X	family of all subsets of X	14
$\setminus$	set-theoretic difference	3
I	unit interval	3
N	set of all natural numbers	1
R	set of all real numbers	1
Z	set of all integers	1
F_σ	F -sigma	93
G_δ	G -delta	93
I^n	unit n -cube	37
I^N	Hilbert cube	37
P^n	projective n -space	44
R^n	Euclidean n -space	37

S^n	unit n -sphere	54
Δ^n	unit n -simplex	41
X^M	Cartesian power	14
f^{-1}	inverse of f	7
$g \circ f$	composition of f and g	8
$f _A$	restriction of f to A	9
$f \times g$	Cartesian product	12
χ_A	characteristic function	10
$f: A \rightarrow B$	function f from A to B	8
$\beta(X)$	Stone-Čech compactification of X	98
$\partial(E)$	boundary of E	21
$\dim(X)$	dimension of X	103
$\text{Cl}(E)$	closure of E	23
$\text{Ext}(E)$	exterior of E	21
$\text{Int}(E)$	interior of E	21

Chapter I: SETS AND FUNCTIONS

In this introductory chapter of the book, we will give an elementary account of sets and functions, with the primary purpose of introducing the notation to be used in the sequel. To save the reader unnecessary effort, this topic will be developed at as naïve a level as possible and with minimal coverage. Among the materials omitted from an earlier manuscript are the basic logical notions, relations, the cardinality of sets, etc. These are so fundamental that we can safely assume that the student has already learned them in some earlier course. Further, we will not indulge in the various forms of the Axiom of Choice and their equivalence. With the single exception in the proof of (III, 2.10), this axiom is used in the book only in its naïve form of allowing an unlimited number of choices.

1. SETS

We will adopt a naïve viewpoint in developing an elementary theory of sets. A *set* is to be thought of intuitively as a collection of objects which are either enumerated or are determined by the possession of some common property. This is not a definition, because the word “collection” is only a synonym for the word “set.” In the text, we will occasionally use synonyms, namely, “aggregate,” “family,” etc. The following examples of sets will be helpful in understanding the intuitive meaning of this undefined term.

- (a) The set AMS of all members of the American Mathematical Society.
- (b) The set MAA of all members of the Mathematical Association of America.
- (c) The set N of all natural numbers, i.e. positive integers.
- (d) The set Z of all integers, positive, zero, or negative.
- (e) The set R of all real numbers.

The symbols for the special sets given in the last three examples will be used throughout the book.

The objects in a set X will be called the *members*, the *elements*, or the *points* of X . These may be concrete things or abstract notions. We shall use the symbol $\in$ to stand for the phrase "is a member of." Thus, the notation

$$x \in X$$

reads that " x is a member of X " or equivalently " x belongs to X ." The negation of $x \in X$ will be denoted by

$$x \notin X.$$

To determine a set is to determine its members. In other words, a set X is determined iff one can tell whether or not any given object x belongs to X . Frequently, the members of a set X are determined by the possession of some common property. For example, if $p(x)$ denotes a given statement relating to the object x , then we write

$$X = \{x \mid p(x)\}$$

to state that X is the set of all objects x for which the statement $p(x)$ holds.

A set X is said to be *empty* iff it has no members. The empty set will be denoted by the symbol $\square$. Thus, $X = \square$ reads that X is empty.

A set X is said to be a *singleton* iff it has one and only one member. If the lone member of a singleton X is x , then we denote

$$X = \{x\}.$$

On logical grounds, it is necessary to distinguish between an object x and the set $\{x\}$. However, as a matter of notational convenience, we will frequently use the same symbol x for an object x and the singleton $\{x\}$ which consists of this object x .

More generally, if $x_1, x_2, \dots, x_n$ are n given objects, then

$$X = \{x_1, x_2, \dots, x_n\}$$

stands for the set X which consists of these objects $x_1, x_2, \dots, x_n$ as members.

Now, let A and B denote two given sets. If every member of A belongs to B , then we say that A is *contained in* B , or equivalently, B *contains* A ; in symbols,

$$A \subset B, \quad B \supset A,$$

where the symbol $\subset$ is called the *inclusion*. In this case, A is said to be

a *subset* of B . Among the sets in the examples (c)–(e) given above, we have

$$N \subset Z \subset R.$$

If $A \subset B$ and $B \subset A$, then we say that A and B are equal; in symbols,

$$A = B.$$

In other words, two sets are equal iff they have the same members. If $A \subset B$ and $A \neq B$, then A is said to be a *proper subset* of B .

The subsets of a given set X are frequently defined by imposing further conditions upon the members of X . For example, if $p(x)$ denotes a given statement relating to the member x of X , then

$$\{x \in X \mid p(x)\}$$

stands for the subset of X which consists of all members x of X for which $p(x)$ holds. In this way, we can define the *unit interval* I of real numbers by the formula:

$$I = \{t \in R \mid 0 \leq t \leq 1\}.$$

There are many ways leading to the formation of new sets from old ones. The following three operations are fundamental. The *union* $A \cup B$ is defined to be the set which consists of those objects x which belong to at least one of the sets A and B . The *intersection* $A \cap B$ is defined to be the set which consists of those objects x which belong to both A and B . The *difference* $A \setminus B$ is defined to be the set which consists of those objects x which belong to A but not to B . These definitions may be stated in the form of the following equations:

$$\begin{aligned} A \cup B &= \{x \mid x \in A \text{ or } x \in B\}, \\ A \cap B &= \{x \mid x \in A \text{ and } x \in B\}, \\ A \setminus B &= \{x \mid x \in A \text{ and } x \notin B\}. \end{aligned}$$

THEOREM 1.1. *For arbitrary sets A , B , C and X , the following laws are valid:*

(1.1.1) *The commutative laws:*

$$\begin{aligned} A \cup B &= B \cup A, \\ A \cap B &= B \cap A. \end{aligned}$$

(1.1.2) *The associative laws:*

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap C, \\ A \cap (B \cup C) &= (A \cap B) \cup C. \end{aligned}$$

(1.1.3) *The distributive laws :*

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C). \end{aligned}$$

(1.1.4) *De Morgan formulae :*

$$\begin{aligned} X \setminus (A \cup B) &= (X \setminus A) \cap (X \setminus B), \\ X \setminus (A \cap B) &= (X \setminus A) \cup (X \setminus B). \end{aligned}$$

The proofs of these are straightforward and hence will be left to the student as exercises. As an illustrative example, we will prove the last equality of the De Morgan formulae as follows.

The proof of an equality of sets usually breaks up into two parts.

(i) *Proof of the inclusion*

$$X \setminus (A \cap B) \subset (X \setminus A) \cup (X \setminus B).$$

Let x be any member of $X \setminus (A \cap B)$. Then, by the definition of the difference, we have $x \in X$ and $x \notin A \cap B$. The latter implies that $x \notin A$ or $x \notin B$. Since $x \in X$, $x \notin A$ implies $x \in X \setminus A$ and $x \notin B$ implies $x \in X \setminus B$. Hence we have $x \in X \setminus A$ or $x \in X \setminus B$; in other words, x must be a member of the set $(X \setminus A) \cup (X \setminus B)$. $\parallel$

(ii) *Proof of the inclusion*

$$(X \setminus A) \cup (X \setminus B) \subset X \setminus (A \cap B).$$

Let x be any member of $(X \setminus A) \cup (X \setminus B)$. Then $x \in X \setminus A$ or $x \in X \setminus B$. If $x \in X \setminus A$, then $x \in X$ and $x \notin A$. Since $x \notin A$ implies $x \notin A \cap B$, it follows that x is in $X \setminus (A \cap B)$. Similarly, one can prove that $x \in X \setminus B$ also implies that x is in $X \setminus (A \cap B)$. $\parallel$

Two sets A and B are said to be *disjoint* iff $A \cap B = \square$; otherwise, they are said to be *overlapping*.

The concept of union and intersection can be generalized to any number of sets as follows. If Φ is a family of sets, then

$$\begin{aligned} \bigcup_{X \in \Phi} X &= \{x \mid x \in X \text{ for some } X \in \Phi\}, \\ \bigcap_{X \in \Phi} X &= \{x \mid x \in X \text{ for each } X \in \Phi\}. \end{aligned}$$

One can verify that the laws in Theorem 1.1 also hold for any number of sets.

If A is a subset of a set X , then the difference $X \setminus A$ will be called the *complement* of A with respect to X . In symbols,

$$\mathcal{C}_X A = X \setminus A.$$

If A is also a subset of some other set Y , then $\mathcal{C}_X A$ and $\mathcal{C}_Y A$ are different sets. If we consider, in a certain situation, only subsets of a fixed set X , then we write $\mathcal{C}A$ instead of $\mathcal{C}_X A$.

EXERCISES

- 1A. Prove all of Theorem 1.1 (including the De Morgan Formulae).
- 1B. Verify the following relations for arbitrary sets A and B :
- $\square \subset A$.
 - $A \subset A$.
 - $A \cup \square = A$.
 - $A \cap \square = \square$.
 - $A \cap A = A = A \cup A$.
 - $A \cap B \subset A \subset A \cup B$.
- 1C. Establish the following propositions for arbitrary sets A, B, C :
- If $A \subset B$ and $B \subset C$, then $A \subset C$.
 - If $A \subset C$ and $B \subset C$, then $A \cup B \subset C$.
 - If $A \supset C$ and $B \supset C$, then $A \cap B \supset C$.
- 1D. Show that the following three statements are equivalent for arbitrary sets A and B :
- $A \subset B$.
 - $A \cup B = B$.
 - $A \cap B = A$.
- 1E. Prove the following properties of subsets of a fixed set X :
- $\mathcal{C}X = \square$.
 - $\mathcal{C}\square = X$.
 - $A \cup \mathcal{C}A = X$.
 - $A \cap \mathcal{C}A = \square$.
 - $\mathcal{C}\mathcal{C}A = A$.
 - $\mathcal{C}(A \cup B) = \mathcal{C}A \cap \mathcal{C}B$.
 - $\mathcal{C}(A \cap B) = \mathcal{C}A \cup \mathcal{C}B$.
 - $\mathcal{C}(A \setminus B) = B \cup \mathcal{C}A$.
 - If $A \cup B = X$ and $A \cap B = \square$, then $B = \mathcal{C}A$.
 - If $A \subset B$, then $\mathcal{C}B \subset \mathcal{C}A$.
- 1F. Verify the following equalities for arbitrary sets A, B, C :
- $A \setminus (A \setminus B) = A \cap B$.
 - $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.
 - $(A \setminus B) \cup (A \setminus C) = A \setminus (B \cap C)$.

- (d) $(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C.$
- (e) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$
- (f) $A \cup (B \setminus A) = A \cup B.$
- (g) $A \cap (B \setminus A) = \square.$

2. FUNCTIONS

Let X and Y be given sets. By a *function* $f: X \rightarrow Y$ from X into Y , we mean a rule which assigns to each member x of X a unique member $f(x)$ of Y .

EXAMPLE 1. Consider the set N of all natural numbers and the set Z_p of all non-negative integers less than a given positive integer p . For any $x \in N$, let us divide x by p and obtain a remainder $f(x)$. This number $f(x)$ is in Z_p . The assignment $x \rightarrow f(x)$ defines a function $f: N \rightarrow Z_p$.

EXAMPLE 2. Consider the equation $y = x^2$. For each real number $x \in R$, $y = x^2$ is also a real number. Hence, the assignment $x \rightarrow x^2$ defines a function $f: R \rightarrow R$ from R into itself. This function is frequently denoted by x^2 ; therefore, $x^2: R \rightarrow R$.

Let $f: X \rightarrow Y$ be a given function. The set X is called the *domain* of the function f and the set Y is called the *range* of f . For each point x of the domain X , the point $f(x)$ of the range Y which is assigned to x by the function f is called the *image* of x under f . Sometimes, $f(x)$ is called the *value* of the function f at the point x .

For any subset A of X , the subset of Y which consists of the points $f(x)$ for all $x \in A$ is called the *image* of A under the function f and will be denoted by $f(A)$; in symbols,

$$f(A) = \{f(x) \in Y \mid x \in A\}.$$

In particular, the image $f(X)$ of the whole domain X of f is simply called the *image* of f and denoted by $\text{Im}(f)$.

PROPOSITION 2.1. For any two subsets A and B of the domain X of a function $f: X \rightarrow Y$, we have

$$(2.1.1) \quad f(A \cup B) = f(A) \cup f(B),$$

$$(2.1.2) \quad f(A \cap B) \subset f(A) \cap f(B).$$

The proofs of (2.1.1) and (2.1.2) are left to the student as exercises;

he can also easily generalize these to any number of subsets of the domain X .

That the two sides of the inclusion (2.1.2) are not always equal can be seen by the following example. Let

$$X = \{a, b\}, \quad A = \{a\}, \quad B = \{b\}, \quad Y = \{y\},$$

and let $f: X \rightarrow Y$ denote the unique function. Then we have

$$f(A \cap B) = f(\emptyset) = \emptyset, \quad f(A) \cap f(B) = Y.$$

If $f(X) = Y$, then we say that $f: X \rightarrow Y$ is a function from X *onto* Y ; frequently, we shall also say that the function $f: X \rightarrow Y$ is *surjective*. Therefore, $f: X \rightarrow Y$ is surjective iff, for every point y in Y , there exists at least one point x in X such that $f(x) = y$. The function in Example 1 is surjective while that in Example 2 is not surjective.

If $f(X)$ consists of a single point y of Y , then we say that $f: X \rightarrow Y$ is a *constant function* from X into Y . If X is nonempty, then, for each $y \in Y$, there is a unique constant function $f_y: X \rightarrow Y$ such that $f_y(X) = y$.

For any subset B of Y , the subset of X which consists of the points $x \in X$ such that $f(x) \in B$ is called the *inverse image* of B under the function $f: X \rightarrow Y$ and will be denoted by $f^{-1}(B)$; in symbols,

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

In particular, if B is a singleton, say $B = \{y\}$, then $f^{-1}(B)$ is called the *inverse image* of the point y under f denoted by $f^{-1}(y)$. Thus, a point $y \in Y$ belongs to the image $f(X)$ of a function f iff $f^{-1}(y)$ is nonempty.

PROPOSITION 2.2. *For any two subsets A and B of the range Y of a function $f: X \rightarrow Y$, we have*

$$(2.2.1) \quad f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B),$$

$$(2.2.2) \quad f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B),$$

$$(2.2.3) \quad f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B).$$

The proofs of (2.2.1)–(2.2.3) are left to the student as exercises; he can also easily generalize the first two equalities to any number of subsets of the range Y .

By a comparison of the propositions (2.1) and (2.2), one will find that the inverse images behave much better than the images. This explains why the notion of inverse images will be used more than that of images.

If A and B are disjoint subsets of Y , then it follows from (2.2.2)

that the inverse images $f^{-1}(A)$ and $f^{-1}(B)$ are also disjoint. In particular, the inverse images of distinct points of Y are disjoint.

A function $f: X \rightarrow Y$ is said to be *one-to-one* or *injective* iff, for every point $y \in Y$, the inverse image $f^{-1}(y)$ is either empty or a singleton. Thus, f is injective iff the images of distinct points of X are distinct. As an example of injective functions, let us consider the case $X \subset Y$. Then, the function $i: X \rightarrow Y$ defined by $i(x) = x \in Y$ for every $x \in X$ is called the *inclusion function* of X into Y . To indicate that $i: X \rightarrow Y$ stands for the inclusion function, we write

$$i: X \subset Y.$$

It is obvious that every inclusion function is injective.

A function $f: X \rightarrow Y$ which is both surjective and injective will be called a *bijective function*. If $f: X \rightarrow Y$ is bijective, then, for every $y \in Y$, the inverse image $f^{-1}(y)$ is always a singleton, i.e. a point in X ; the assignment $y \rightarrow f^{-1}(y)$ defines a function $g: Y \rightarrow X$, which is called the *inverse function* of f and may be denoted by

$$f^{-1}: Y \rightarrow X.$$

One can easily see that f^{-1} is also bijective. As an example of bijective functions, we mention the inclusion function $i: X \subset X$ of X into itself. This special inclusion function i will be called the *identity function* on X . For this special case, we have $i^{-1} = i$.

Two functions f and g are said to be *composable* iff the range of f is equal to the domain of g , i.e.

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

In this case, we define a function $\phi: X \rightarrow Z$ by assigning to each point x of X the point

$$\phi(x) = g[f(x)]$$

of the set Z . This function ϕ is called the *composition* of f and g denoted by

$$\phi = g \circ f: X \rightarrow Z.$$

PROPOSITION 2.3. *If $\phi = g \circ f$ denotes the composition of the functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, then we have:*

$$(2.3.1) \quad \phi(A) = g[f(A)] \quad \text{for each } A \subset X;$$

$$(2.3.2) \quad \phi^{-1}(C) = f^{-1}[g^{-1}(C)] \quad \text{for each } C \subset Z.$$

The proofs of (2.3.1) and (2.3.2) are left to the student as exercises.

It follows from (2.3) that the composition of surjective functions is surjective and that the composition of injective functions is injective. As partial converse of this, we have the following proposition which will be useful in the sequel.

PROPOSITION 2.4. *If $\phi = g \circ f$ denotes the composition of the functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, then the following statements are true:*

(2.4.1) *If ϕ is surjective, then so is g .*

(2.4.2) *If ϕ is injective, then so is f .*

Proof. Assume that ϕ is surjective. Then $\phi(X) = Z$ by definition. According to (2.3.1), we have

$$Z = \phi(X) = g[f(X)] \subset g(Y) \subset Z.$$

It follows that $g(Y) = Z$ and hence g is surjective. This proves (2.4.1).

Next, assume that ϕ is injective. Let a and b be any two points in X such that $f(a) = f(b)$. Then we have

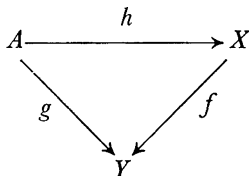
$$\phi(a) = g[f(a)] = g[f(b)] = \phi(b).$$

Since ϕ is injective, it follows that $a = b$. This proves (2.4.2). $\parallel$

Let $f: X \rightarrow Y$ be a given function and A be a subset of X . Define a function $g: A \rightarrow Y$ by taking $g(x) = f(x)$ for every $x \in A$. This function g will be called the *restriction* of the given function f to the subset A ; in symbols

$$g = f|A.$$

If $g = f|A$, then the function $f: X \rightarrow Y$ is said to be an *extension* of the function $g: A \rightarrow Y$ over the set X . In this case, we obtain a triangle



of functions, where $h: A \subset X$ stands for the inclusion function. The relation $g = f|A$ is equivalent to the commutativity of the triangle, i.e., $g = f \circ h$.

While there is only one restriction of a given function $f: X \rightarrow Y$ to a given subset A of X , the extensions of a given function $g: A \rightarrow Y$ over a set

X which contains A are usually numerous. For example, let y be an arbitrary point in Y ; the function $e_y: X \rightarrow Y$ defined by

$$e_y(x) = \begin{cases} g(x), & (\text{if } x \in A), \\ y, & (\text{if } x \in X \setminus A), \end{cases}$$

is an extension of the given function $g: A \rightarrow Y$ over the set X .

The definition of the function $e_y: X \rightarrow Y$ given above is a special case of the construction of *combined functions* which will be frequently used in the sequel. Let F be a given family of subsets of a set X . Assume that F covers X , that is to say, X is equal to the union of the sets in F . Assume that, for each $A \in F$, there has been given a function $f_A: A \rightarrow Y$. Thus, we obtain a family

$$\Phi = \{f_A \mid A \in F\}$$

of functions indexed by the members of the family F . The family Φ of functions is said to be *combinable* iff, for any two sets $A, B \in F$, the functions $f_A: A \rightarrow Y$ and $f_B: B \rightarrow Y$ agree on the intersection $A \cap B$, i.e.

$$f_A \mid A \cap B = f_B \mid A \cap B.$$

If the family Φ of functions is combinable, then Φ defines uniquely a function $f: X \rightarrow Y$ given by taking $f(x) = f_A(x)$ if $x \in A \in F$. This function f will be called the *combined function* of the family Φ of functions.

To conclude the present section, we will give a few more examples of functions of special kinds.

EXAMPLE 3. A function $f: N \rightarrow X$ from the set N of natural numbers into a given set X is called a *sequence* (of points) in X . For each $n \in N$, the image $x_n = f(n)$ is called the *n th term* of the sequence f . Customarily, the sequence f can be written in the form

$$f = \{x_1, x_2, x_3, \dots, x_n, \dots\}.$$

In particular, if X is the set R of real numbers, then f is called a sequence of real numbers; if X is the set Z of integers, then f is called a sequence of integers.

EXAMPLE 4. Let X be a given set. For an arbitrary subset A of X , define a function $\chi_A: X \rightarrow R$ by taking

$$\chi_A(x) = \begin{cases} 1, & (\text{if } x \in A), \\ 0, & (\text{if } x \in X \setminus A). \end{cases}$$

This function χ_A is called the *characteristic function* of the subset A in X .

EXAMPLE 5. Let 2^X denote the set of all subsets of a given set X . Consider an arbitrary function $f: M \rightarrow 2^X$ from a set M into 2^X . For each element $\alpha \in M$, the image $E_\alpha = f(\alpha)$ is a subset of X . Customarily, the function f can be written in the form

$$f = \{E_\alpha \subset X \mid \alpha \in M\}$$

and is called an *indexed family* of sets with M as the *set of indices*. In particular, if M is the set N of natural numbers, then f is called a *sequence of sets*.

EXERCISES

2A. Prove the propositions (2.1), (2.2), and (2.3).

2B. Establish the following relations for any function $f: X \rightarrow Y$ with $A \subset X$ and $B \subset Y$:

- (a) $f^{-1}(f(A)) \supset A$.
- (b) $f(f^{-1}(B)) \subset B$.
- (c) $f(X \setminus A) \supset f(X) \setminus f(A)$.
- (d) $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$.
- (e) $f(A \cap f^{-1}B) = f(A) \cap B$.

2C. Prove that a function $f: X \rightarrow Y$ is bijective iff there exist two functions $g, h: Y \rightarrow X$ such that the compositions $g \circ f$ and $f \circ h$ are the identity functions on X and Y respectively. In this case $g = f^{-1} = h$.

2D. Prove that the composition is associative, i.e., for arbitrary functions $f: X \rightarrow Y$, $g: Y \rightarrow Z$, and $h: Z \rightarrow W$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Hence, we may denote this composed function by $h \circ g \circ f$.

2E. Verify the following equalities for the characteristic functions of subsets of X at an arbitrary point x of X :

- (a) $\chi_{A \cap B}(x) = \chi_A(x)\chi_B(x)$.
- (b) $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$.
- (c) $\chi_{A \setminus B}(x) = \chi_A(x)[1 - \chi_B(x)]$.

2F. If $f: X \rightarrow Y$ is a function and $\{E_\alpha \mid \alpha \in M\}$ is an indexed family of subsets of Y , then the following two equalities hold:

- (a) $f^{-1}(\bigcup_{\alpha \in M} E_\alpha) = \bigcup_{\alpha \in M} f^{-1}(E_\alpha)$.
- (b) $f^{-1}(\bigcap_{\alpha \in M} E_\alpha) = \bigcap_{\alpha \in M} f^{-1}(E_\alpha)$.

3. CARTESIAN PRODUCTS

Let us consider an arbitrarily given indexed family of sets

$$\mathcal{F} = \{X_\mu \mid \mu \in M\}$$

and denote by X the union of the sets X_μ for all $\mu \in M$.

By the *Cartesian product* of the family $\mathcal{F}$ of sets, we mean the set Φ of all functions

$$f: M \rightarrow X$$

such that $f(\mu) \in X_\mu$ for every $\mu \in M$. The Cartesian product of the family $\mathcal{F}$ is denoted by

$$\Phi = \prod_{\mu \in M} X_\mu.$$

In particular, if M consists of the first n natural numbers, then a point f in Φ is essentially an ordered n -tuple $(x_1, \dots, x_n)$ with $x_i = f(i)$ for each $i = 1, \dots, n$. In this case, the Cartesian product of the family $\mathcal{F}$ is denoted by

$$\Phi = X_1 \times \cdots \times X_n.$$

If $X_\mu = \square$ for some $\mu \in M$, then one can easily see that the Cartesian product Φ is empty; otherwise, we have $\Phi \neq \square$. Hereafter, we will always assume that $X_\mu \neq \square$ for every $\mu \in M$.

For each $\mu \in M$, consider the function

$$p_\mu: \Phi \rightarrow X_\mu$$

defined by $p_\mu(f) = f(\mu)$ for every $f \in \Phi$. By the axiom of choice, p_μ is surjective for every $\mu \in M$. We will call p_μ the *projection* of the Cartesian product onto its μ -th coordinate set.

If each member X_μ of the family $\mathcal{F}$ is equal to a given set X , then the Cartesian product Φ of the family $\mathcal{F}$ will be called the *Mth Cartesian power* of the set X denoted by

$$\Phi = X^M.$$

Hence, X^M is the set of all functions from M into X . In particular, if M consists of the first n natural numbers $1, \dots, n$, Φ is called the *nth Cartesian power* of the set X ; in symbols,

$$\Phi = X^n.$$

Thus, X^n is the set of all n -tuples $(x_1, \dots, x_n)$ with $x_i \in X$ for every $i = 1, \dots, n$.

Assume $M \neq \square$ and consider the function

$$d: X \rightarrow X^M$$

defined by taking $d(x) \in X^M$ to be the constant function

$$[d(x)](M) = x$$

for every point $x \in X$. Obviously, the function d is injective. It is called the *diagonal injection* of the set X into the Cartesian power X^M .

Next, let us consider an arbitrarily given indexed family

$$\mathcal{H} = \{h_\mu: X_\mu \rightarrow Y_\mu \mid \mu \in M\}$$

of functions. Denote

$$X = \bigcup_{\mu \in M} X_\mu, \quad Y = \bigcup_{\mu \in M} Y_\mu, \\ \Phi = \prod_{\mu \in M} X_\mu, \quad \Psi = \prod_{\mu \in M} Y_\mu.$$

Define a function

$$H: \Phi \rightarrow \Psi$$

as follows. For an arbitrary point $f \in \Phi$, $H(f) \in \Psi$ is defined to be the function $H(f): M \rightarrow Y$ given by

$$[H(f)](\mu) = h_\mu[f(\mu)]$$

for every $\mu \in M$. To justify this definition, we observe that, for each $\mu \in M$, $f(\mu)$ is a point of X_μ and hence $h_\mu[f(\mu)]$ is a well-defined point of Y_μ . This function $H: \Phi \rightarrow \Psi$ will be called the *Cartesian product* of the family $\mathcal{H}$ of functions and will be denoted by

$$H = \prod_{\mu \in M} h_\mu.$$

In particular, if M consists of the first n natural numbers, then the Cartesian product of the family $\mathcal{H}$ is denoted by

$$H = h_1 \times \cdots \times h_n.$$

If $X_\mu = X$ for every $\mu \in M$, then $\Phi = X^M$ and the diagonal injection d is defined. The composed function

$$h = H \circ d: X \rightarrow \Psi$$

of the functions d and H in the following diagram

$$X \xrightarrow{d} X^M \xrightarrow{H} \Psi$$

will be called the *restricted Cartesian product* of the family

$$\mathcal{H} = \{h_\mu: X \rightarrow Y_\mu \mid \mu \in M\}.$$

When there is no risk of ambiguity, this function h will also be called the Cartesian product of the family $\mathcal{H}$ and also be denoted by

$$h = \prod_{\mu \in M} h_{\mu}.$$

EXERCISES

3A. Show that if $A \subset X$ and $B \subset Y$, then

$$(a) A \times B \subset X \times Y,$$

$$(b) (X \times Y) \setminus (A \times B) = [(X \setminus A) \times Y] \cup [X \times (Y \setminus B)].$$

3B. Show that if $A \subset X$, $B \subset Y$, $C \subset X$, and $D \subset Y$, then

$$(a) (A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D),$$

$$(b) (A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D).$$

Give an example showing that the two members of (b) fail to be equal.

3C. Consider the function $\theta: X^2 \rightarrow X^2$ defined by $\theta(a, b) = (b, a)$ for every point (a, b) of the Cartesian square X^2 . Verify

$$\theta \circ d = d,$$

where $d: X \rightarrow X^2$ denotes the diagonal injection. Generalize this fact to an arbitrary Cartesian power X^M .

3D. Consider the Cartesian product Φ of an arbitrarily given family $\mathcal{F} = \{X_{\mu} \mid \mu \in M\}$ of sets and its projections

$$p_{\mu}: \Phi \rightarrow X_{\mu}, \quad (\mu \in M).$$

Prove that the restricted Cartesian product of the family $\{p_{\mu} \mid \mu \in M\}$ is the identity function on Φ .

3E. Consider the diagonal injection $d: X \rightarrow X^M$ and the projections $p_{\mu}: X^M \rightarrow X$, $(\mu \in M)$, of a Cartesian power X^M . Prove that the composed function

$$p_{\mu} \circ d: X \rightarrow X$$

is the identity function on X for every $\mu \in M$.

3F. Assume that the set X consists of the integers 0 and 1. Define a function

$$\beta: 2^M \rightarrow X^M$$

from the set 2^M of all subsets of the set M into the Cartesian power X^M as follows: for each subset S of M , take $\beta(S)$ to be the characteristic function of the set S , i.e.

$$\beta(S) = \chi_S: M \rightarrow X.$$

Prove that the function β is bijective. This explains the meaning of the classical notation 2^M for the set of all subsets of a given set M .

3G. Define a function

$$e : X^M \times M \rightarrow X$$

by taking $e(f, \mu) = f(\mu)$ for each $\mu \in M$ and each $f \in X^M$. This function is called the *evaluation* of the Cartesian power X^M . For any given $\mu \in M$, verify

$$p_\mu(f) = e(f, \mu)$$

for every $f \in X^M$. Hence, e can be considered as the projections collected all together.

Chapter II: SPACES AND MAPS

Our main objective in this chapter is the introduction of spaces and maps. We will adopt as a definite principle that the topology of a space means the totality of its open sets and nothing else. While there are various ways to determine the topology of a space, we shall not regard these as the topology itself. Most of the chapter will be devoted to the methods of constructing new spaces from old ones by forming subspaces, topological sums, topological products, or quotient spaces, and also of constructing new maps from old ones by means of composition or combination.

1. TOPOLOGICAL SPACES

Let X be a given *set* of objects called the *points* of X .

A *topology* in X is a nonempty collection

$$\tau \subset 2^X$$

of subsets of X called *open sets* satisfying the following four axioms:

(OS1) *The empty set $\square$ is open.*

(OS2) *The set X itself is open.*

(OS3) *The union of any family of open sets is open.*

(OS4) *The intersection of any two (and hence of any finite number of) open sets is open.*

A set X is said to be *topologized* if a topology τ has been given in X . A topologized set X will be called a *topological space* or simply a *space* and the topology τ in X is called the *topology of the space* X . The open sets U in τ will be called the *open sets of the space* X .

EXAMPLES OF SPACES:

(1) *Discrete spaces.* Let X be any set. The collection of *all* subsets of X obviously satisfies the axioms for open sets and hence is a topology in X called the *discrete topology* in X . If X is topologized by its discrete topology, it is called a *discrete space*.

(2) *Indiscrete spaces.* Let X be any set. The collection $\{\emptyset, X\}$ satisfies the axioms for open sets and hence is a topology in X called the *indiscrete* (or *trivial*) topology in X . If X is topologized by its indiscrete topology, it is called an *indiscrete space*.

(3) *The real line.* Let R denote the set of all real numbers. Define a collection τ of subsets of R as follows: A subset U of R is in τ iff, for an arbitrarily given point $u \in U$, there exists a positive real number δ_u such that a real number x is in U if $|x - u| < \delta_u$. It can be verified that this collection τ forms a topology in R which will be referred to as the *usual topology* in R . If R is topologized by its usual topology, it is called the (topological) *real line*.

Since topologies in a set X are special collections of subsets of X , we can compare topologies in the same set X by means of the inclusion relation. Let σ and τ be any two topologies in X . If $\sigma \subset \tau$, then we shall say that σ is *smaller* than τ and τ is *larger* than σ . It is also said that σ is *coarser* than τ and τ is *finer* than σ . Unfortunately, the more familiar terms "stronger" and "weaker" are used in contradictory senses in algebraic topology and functional analysis: If $\sigma \subset \tau$, in algebraic topology σ is said to be stronger than τ , while in functional analysis σ is said to be weaker than τ . On the other hand, τ is said to be weaker than σ in algebraic topology and τ is said to be stronger than σ in functional analysis.

A *basis* of a topology τ in X is a subcollection β of τ such that every open set U in τ is a union of some open sets in β . In other words, for every $U \in \tau$ and each point $x \in U$, there is a $V \in \beta$ such that

$$x \in V \subset U.$$

The open sets of a given basis β will be called the *basic open sets* of the space X .

A *sub-basis* of a topology τ in X is a subcollection σ of τ such that the finite intersections (= the intersections of the finite families) of open sets in σ form a basis of τ . In other words, for every $U \in \tau$ and each $x \in U$, there are a finite number of open sets in σ , say $W_1, \dots, W_n$, such that

$$x \in W_1 \cap \dots \cap W_n \subset U.$$

The open sets in a given sub-basis σ will be called the *sub-basic open sets* of the space X .

It is obvious from the definitions that the topology τ of a space X is completely determined by any given basis or sub-basis of τ .

EXAMPLE OF BASIS AND SUB-BASIS. Let us consider the real line R as

a space defined in Example 3 above. For any two distinct real numbers a and b , $a < b$, the set

$$(a, b) = \{x \in R \mid a < x < b\}$$

is called an *open interval* of R . It follows from the definition of open sets in R that the collection β of all open intervals of R forms a basis of the real line R . On the other hand, for each real number a , the sets

$$L_a = \{x \in R \mid x < a\}, \quad R_a = \{x \in R \mid x > a\}$$

are called the *open half lines* determined by the real number a . Since

$$(a, b) = R_a \cap L_b,$$

it follows that the collection σ of all open half lines forms a sub-basis of R .

A space X is said to satisfy the *second axiom of countability* iff its topology has a countable basis. As we shall see further on, such spaces have many pleasant properties. From the definition of sub-basis, one can easily see that a space X satisfies the second axiom of countability iff its topology has a countable sub-basis.

Let X be a given space and p be a given point in X . A set $N \subset X$ is said to be a *neighborhood* of the point p in the space X iff there exists an open set U of X such that

$$p \in U \subset N.$$

Hence a neighborhood of a point p in a space X is a set in X containing not only the point p itself but also some open set of X which contains p .

PROPOSITION 1.1. *A set U of a space X is open iff U contains a neighborhood of each of its points.*

Proof: Necessity. Assume that U is an open set of a space X . Then, by the definition of neighborhood of points in X , U is a neighborhood of each point of U . This implies the necessity.

Sufficiency. Assume that U is a set in X such that U contains a neighborhood of each of its points. Let p be an arbitrary point of U . Then, by assumption, there exists a neighborhood N_p of the point p in X which is contained in U . By the definition of neighborhoods, there exists an open set U_p of the space such that

$$p \in U_p \subset N_p.$$

By (OS3), the union

$$U = \bigcup_{p \in U} p \subset \bigcup_{p \in U} U_p \subset U.$$

Hence, we obtain:

$$\bigcup_{p \in U} U_p = U.$$

Since U_p is open for each p , this implies the sufficiency. $\parallel$

For an arbitrarily given point p in a space X , the family of all neighborhoods of p in X will be called the *neighborhood system* of the point p in the space X .

Properties of neighborhood systems are given by the following proposition.

PROPOSITION 1.2. *If $\mathcal{N}$ denotes the neighborhood system of a point p in a space X , then finite intersections of members of $\mathcal{N}$ belong to $\mathcal{N}$ and each set in X which contains a member of $\mathcal{N}$ belongs to $\mathcal{N}$.*

Proof. Let $N_1, \dots, N_r$ be given neighborhoods of p in X . Then, by definition, there exist open sets $U_1, \dots, U_r$ in X such that

$$p \in U_i \subseteq N_i, \quad (i = 1, 2, \dots, r).$$

Let $N = N_1 \cap \dots \cap N_r$ and $U = U_1 \cap \dots \cap U_r$. Then we have $p \in U \subseteq N$. By (OS4), U is an open set. Hence, N is a neighborhood of p in X .

Next, let L be any neighborhood of p in X and M be a set in X which contains L . By the definition of a neighborhood, there exists an open set V of X with $p \in V \subseteq L$. Since $L \subseteq M$, it follows that $p \in V \subseteq M$. Hence, M is a neighborhood of p in X . $\parallel$

By a *local basis*, or a *neighborhood basis*, of a space X at a point $p \in X$, we mean a collection β of neighborhoods of p in X such that every neighborhood of p in X contains a member of β . The members of a given local basis β of X at a point $p \in X$ will be called the *basic neighborhoods* of p in X .

EXAMPLE OF LOCAL BASIS. Consider the real line R and any $p \in R$. For each positive real number $\delta > 0$, the set

$$N_\delta(p) = \{x \in R \mid |x - p| < \delta\}$$

is called the δ -neighborhood of p in R . Let $\delta_1, \delta_2, \dots, \delta_i, \dots$ be any sequence of positive real numbers converging to 0; then the collection $\{N_{\delta_i}(p) \mid i = 1, 2, \dots\}$ is a local basis of R at p .

A space X is said to satisfy the *first axiom of countability* iff it has a countable local basis at each of its points. For examples, the real line R and all discrete spaces satisfy the first axiom of countability.

Since the family of all open sets in a given basis of a space X containing a given point $p \in X$ is obviously a local basis of X at the point p , we get the following proposition.

PROPOSITION 1.3. *If a space X satisfies the second axiom of countability, then it satisfies also the first axiom of countability.*

The converse of (1.3) is false; for example, every noncountable discrete space fails to satisfy the second axiom of countability.

EXERCISES

1A. Find all topologies on a set of three elements.

1B. Prove that, for any family $\mathcal{F}$ of topologies in a set X , there is a unique largest topology $\text{Inf } \mathcal{F}$ in X which is smaller than each member of $\mathcal{F}$, and a unique smallest topology $\text{Sup } \mathcal{F}$ in X which is larger than each member of $\mathcal{F}$. Furthermore, prove that

$$\text{Inf } \mathcal{F} = \bigcap \{ \tau \mid \tau \in \mathcal{F} \}.$$

1C. Define a collection σ of subsets of R as follows: a set W of R is in σ iff either $W = \square$ or $R \setminus W$ consists of at most a finite number of real numbers. Verify that σ is a topology in R which is properly smaller than the usual topology in R .

1D. Let β denote the collection of all half-open intervals $[a, b)$, which consist of all real numbers x with $a \leq x < b$, where a and b are real numbers. Let $\mathcal{F}$ denote the family of topologies in X each containing β and let

$$\eta = \text{Inf } \mathcal{F}.$$

Then η is called the *half-open interval topology* of R . Prove that η is different from both the usual topology τ and the topology σ in (1C).

1E. Prove that the half lines L_a, R_a for all rational real numbers a constitute a sub-basis of the real line R and hence the real line R satisfies the second axiom of countability.

1F. Prove that if the topology τ of a space X has a countable basis, then each basis of τ contains a countable subcollection which is also a basis of τ .

1G. Let σ be any given family of subsets of a set X such that the union of all members of σ is the whole set X . Prove that σ is a sub-basis of the smallest topology τ in X which contains σ .

1H. Let $\mathcal{N}$ denote a function which assigns to each point x of a set X a nonempty family $\mathcal{N}_x$ of subsets of X which satisfies the following three conditions:

- (i) If $N \in \mathcal{N}_x$, then $x \in N$.
- (ii) If M and N are members of $\mathcal{N}_x$, then $M \cap N$ belongs to $\mathcal{N}_x$.
- (iii) If $M \in \mathcal{N}_x$ and $M \subset N \subset X$, then $N \in \mathcal{N}_x$.

Prove that the collection τ of subsets of X defined by

$$\tau = \{U \subset X \mid x \in U \Rightarrow U \in \mathcal{N}_x\}$$

satisfies (OS1-4) and hence is a topology in X . Furthermore, $\mathcal{N}_x$ is precisely the neighborhood system of x in the space X iff it satisfies the following conditions:

- (iv) If $N \in \mathcal{N}_x$, then there is an $M \in \mathcal{N}_x$ such that $M \subset N$ and $M \in \mathcal{N}_y$ for every $y \in M$.

2. SETS IN A SPACE

Let X be an arbitrarily given space. Consider a set E in X and a point p of X . We will first study the relation between p and E with respect to the topology of the given space X .

The point p is said to be an *interior point* of the set E provided that there exists a neighborhood N of p in X contained in E . The point p is said to be an *exterior point* of E if there exists a neighborhood N of p in X which contains no point of E . Finally, the point p is said to be a *boundary point*, or *frontier point*, of E in case every neighborhood N of p in X contains at least one point in E and at least one point not in E .

The set $\text{Int } (E)$ of all interior points of E is called the *interior* of E . The set $\text{Ext } (E)$ of all exterior points of E is called the *exterior* of E . The set $\partial(E)$ of all boundary points of E is called the *boundary*, or *frontier*, of the set E in the space X . Obviously, we have

$$\begin{aligned}\text{Int } (E) &\subset E, \\ \text{Ext } (E) \cap E &= \square;\end{aligned}$$

and a boundary point of E may be in E and may be not in E .

PROPOSITION 2.1. *The interior $\text{Int } (E)$ of a set E in a space X is the largest open set of X contained in E .*

Proof. Let U be an arbitrary open set of X contained in E . We are going to show that $U \subset \text{Int } (E)$. For this purpose, let p be any point in U . By definition, U is a neighborhood of p in X . Since $U \subset E$, it follows that p is an interior point of E . This implies $U \subset \text{Int } (E)$.

Next, for each interior point p of E , choose a neighborhood N_p of p in X such that $p \in N_p \subset E$. By definition of neighborhoods, there exists an open set U_p of X such that $p \in U_p \subset N_p$. Let U denote the union of the open sets U_p for all interior points p of E . This implies that $\text{Int}(E) \subset U$. Since U is an open set of X according to (OS3), it follows from the first part of the proof that $U \subset \text{Int}(E)$. Hence, $\text{Int}(E) = U$ is an open set of X .

Combining both parts of the proof, we conclude that $\text{Int}(E)$ is open and contains every open set of X which is contained in E . $\parallel$

As an immediate consequence of (2.1), we have the following corollary which gives the geometrical significance of the terminology of open sets.

COROLLARY 2.2. *A set E in a space X is open iff E contains none of its boundary points, in symbols, $E \cap \partial(E) = \square$.*

On the other extreme, we introduce the closed sets to the reader as follows.

A set E in a space X is said to be *closed* iff E contains all of its boundary points, in symbols $\partial(E) \subset E$. Note that a set which contains some but not all of its boundary points is neither open nor closed.

PROPOSITION 2.3. *A set E in a space X is closed iff its complement $X \setminus E$ is open.*

Proof. It follows immediately from the definition of boundary points that

$$\partial(E) = \partial(X \setminus E).$$

To prove the necessity, let us assume that E is closed. Then, according to definition, $\partial(E) \subset E$. This implies that $X \setminus E$ contains none of its boundary points. Hence $X \setminus E$ is open according to (2.2).

To prove the sufficiency, let us assume that $X \setminus E$ is open. By (2.2), $\partial(X \setminus E)$ must be contained in E . Hence, $\partial(E) \subset E$ and E is closed by definition. $\parallel$

The preceding proposition establishes duality between the closed sets and open sets of a space by means of taking complements. Because of this duality, one can easily deduce the properties of closed sets dual to those of open sets. For example, the following proposition can be established in this way.

PROPOSITION 2.4. *The closed sets of a space X satisfy the following four conditions:*

(CS1) The empty set $\square$ is closed.

(CS2) The set X itself is closed.

(CS3) The intersection of any family of closed sets is closed.

(CS4) The union of any two (and hence of any finite number of) closed sets is closed.

Dual to (2.1), we will define the closure of a set E in a space X as follows.

By the *closure* $\text{Cl } (E)$ of a set E in a space X , we mean the smallest closed set of X which contains E . By (CS3), $\text{Cl } (E)$ is equal to the intersection of all closed sets of X containing E . Frequently, we will also use the classical notation $\bar{E}$ for the closure of E in X .

PROPOSITION 2.5. For any given set E in a space X , we have

$$\text{Cl } (E) = E \cup \partial(E) = \text{Int } (E) \cup \partial(E).$$

Proof. From the relations $\text{Int } (E) \subset E$ and $\text{Ext } (E) \subset X \setminus E$, we deduce

$$E \cup \partial(E) = \text{Int } (E) \cup \partial(E) = X \setminus \text{Ext } (E).$$

Since $\text{Ext } (E) = \text{Int } (X \setminus E)$ is open, it follows that $E \cup \partial(E)$ is closed and hence

$$\text{Cl } (E) \subset E \cup \partial(E).$$

On the other hand, let F be any closed set of X containing E . Then, the set $X \setminus F$ is open and is contained in $X \setminus E$. By (2.1), we have

$$X \setminus F \subset \text{Int } (X \setminus E) = \text{Ext } (E).$$

This implies that $F \supset X \setminus \text{Ext } (E)$. In particular, we have

$$\text{Cl } (E) \supset E \cup \partial(E).$$

This completes the proof of (2.5). $\square$

The duality between $\text{Int } (E)$ and $\text{Cl } (E)$ is clarified by the following proposition which is an immediate consequence of (2.1) and the definition of $\text{Cl } (E)$.

PROPOSITION 2.6. For any given set E in a space X , we have:

- (i) E is open iff $\text{Int } (E) = E$;
- (ii) E is closed iff $\text{Cl } (E) = E$.

A characterization of the points in $\text{Cl } (E)$ is given by the following proposition.

PROPOSITION 2.7. *A point p is in the closure $\text{Cl}(E)$ of a set E in a space X iff every neighborhood N of p in X meets the set E .*

Proof: Necessity. Assume that $p \in \text{Cl}(E)$. Since $\text{Cl}(E) = E \cup \partial(E)$ by (2.5), we must have $p \in E$ or $p \in \partial(E)$. If $p \in E$, then $N \cap E$ contains the point p ; if $p \in \partial(E)$, then it follows from the definition of $\partial(E)$ that N contains at least one point of E . Hence $N \cap E \neq \emptyset$ and the necessity is established.

Sufficiency. Assume that every neighborhood of p meets E . We have to prove that $p \in \text{Cl}(E) = E \cup \partial(E)$. This is obvious if $p \in E$. Hence, we may assume that $p \notin E$. Then every neighborhood of p contains also a point not in E . By definition, $p \in \partial(E) \subset \text{Cl}(E)$. This proves the sufficiency. $\square$

If p is a point in $\text{Cl}(E)$ but not in E , then every neighborhood N of p in X contains at least one point of E other than p . This suggests the following definition.

A point p of a space X is said to be an *accumulation point* (or *cluster point*, or *limit point*) of a set E in X iff every neighborhood N of p in X contains at least one point of E other than p . The set E' of all accumulation points of E in X is called the *derived set* of E in X . Then, we have

$$\text{Cl}(E) = E \cup E'$$

and that a set E in X is closed iff it contains all of its accumulation points.

A set E in a space X is said to be *dense* in X iff $\text{Cl}(E) = X$.

PROPOSITION 2.8. *A set E in a space X is dense in X iff the topology of the space X has a basis such that every nonempty basic open set meets E .*

Proof: Necessity. It suffices to show that every nonempty open set U of X meets E . For this purpose choose an arbitrary point $p \in U$. Since E is dense, p is a point of $\text{Cl}(E)$. Since U is a neighborhood of p , it follows from (2.7) that U meets E .

Sufficiency. Assume that the topology of the space X has a basis β such that every nonempty basic open set meets E . Let p be an arbitrary point in X and N be any neighborhood of p in X . By the definition of neighborhood and basis, there exists a basic open set V such that $p \in V \subset N$. By assumption, V meets E and hence N also meets E . Hence $p \in \text{Cl}(E)$ according to (2.7). Since p is arbitrary, we have $\text{Cl}(E) = X$. $\square$

As an immediate consequence of (2.8), the set Q of all rational numbers is dense in the real line R . In fact, every open interval of real

numbers contains a rational number and the collection of open intervals is a basis of the usual topology of R .

A space X is said to be *separable* iff it contains a countable subset which is dense in X . For example, the real line R is separable, because the subset Q of all rational numbers is countable and is dense in R .

PROPOSITION 2.9. *Every space which satisfies the second axiom of countability is separable.*

Proof. Let X be a space satisfying the second axiom of countability. Then, the topology τ of X admits a countable basis β . In each non-empty basic open set $V \in \beta$, let us pick a point $x_V \in V$. Let E denote the set of all points x_V for all $V \in \beta$. Then E must be countable. By (2.8), E is dense in X . Hence X is separable. ||

The converse of (2.9) is false. In fact, a separable space may violate the second axiom of countability. For example, let us consider the set R of real numbers with the topology σ defined in Exercise 1C; that is to say, a set W of R is in σ iff either $W = \square$ or $R \setminus W$ is finite. In this space R topologized by σ , every infinite subset of R is dense in R . In particular, the set N of positive integers is dense in R ; hence, R is separable. On the other hand, we will prove that this space R does not satisfy the second axiom of countability. For this purpose, let us assume that β were a countable basis of the topology σ and deduce a contradiction as follows. Let $p \in R$ and consider the intersection of all $V \in \beta$ such that $p \in V$. We assert that this intersection is equal to the singleton set $\{p\}$. In fact, let $q \neq p$. Then $R \setminus \{q\}$ is in σ . There exists a $V \in \beta$ such that $p \in V \subset R \setminus \{q\}$. Hence, q is not in the intersection. This proves our assertion that the intersection is the singleton set $\{p\}$. Now, on one hand, the complement of this countable intersection is the union of a countable number of finite sets and hence countable. On the other hand, this complement contains all real numbers except p and hence is not countable. This contradiction completes the proof.

EXERCISES

2A. Let A and B be any two sets in a space X . Prove the following properties of the operation of forming the interior of a set:

- $\text{Int} [\text{Int} (A)] = \text{Int} (A)$;
- $\text{Int} (A \cup B) \supset \text{Int} (A) \cup \text{Int} (B)$;
- $\text{Int} (A \cap B) = \text{Int} (A) \cap \text{Int} (B)$.

In the case of (b), give an example of two sets A and B in the real line R such that $\text{Int}(A \cup B)$ is not equal to $\text{Int}(A) \cup \text{Int}(B)$.

2B. Let A and B be any two sets in a space X . Prove the following properties of the operation of forming the closure of a set:

- (a) $\text{Cl}[\text{Cl}(A)] = \text{Cl}(A)$;
- (b) $\text{Cl}[A \cup B] = \text{Cl}(A) \cup \text{Cl}(B)$;
- (c) $\text{Cl}[A \cap B] \subset \text{Cl}(A) \cap \text{Cl}(B)$.

In the case of (c), give an example of two sets A and B in the real line R such that $\text{Cl}(A \cap B)$ is not equal to $\text{Cl}(A) \cap \text{Cl}(B)$.

2C. For an arbitrary set E in a space X , establish the following relations between interior and closure:

- (a) $\text{Int}(X \setminus E) = X \setminus \text{Cl}(E)$;
- (b) $\text{Cl}(X \setminus E) = X \setminus \text{Int}(E)$.

2D. By an *interior operator* on X , we mean a function ϕ which assigns to each subset E of X a subset $\phi(E)$ of X such that the following four conditions are satisfied:

- (IO1) $\phi(X) = X$;
- (IO2) $\phi(E) \subset E$ for each $E \subset X$;
- (IO3) $\phi[\phi(E)] = \phi(E)$ for each $E \subset X$;
- (IO4) $\phi(A \cap B) = \phi(A) \cap \phi(B)$ for each $A \subset X$ and each $B \subset X$.

Let ϕ be a given interior operator on X and let τ denote the collection of all subsets E of X such that $\phi(E) = E$. Verify that τ is a topology in X . Prove that, for each $E \subset X$, $\phi(E)$ is the interior of E in the space X with τ as topology.

2E. By a *closure operator* on X , we mean a function ψ which assigns to each subset E of X a subset $\psi(E)$ of X such that the following four conditions are satisfied:

- (CO1) $\psi(\emptyset) = \emptyset$;
- (CO2) $E \subset \psi(E)$ for each $E \subset X$;
- (CO3) $\psi[\psi(E)] = \psi(E)$ for each $E \subset X$;
- (CO4) $\psi[A \cup B] = \psi(A) \cup \psi(B)$ for each $A \subset X$ and each $B \subset X$.

Let ψ be a given closure operator on X and let τ denote the collection of all subsets E of X such that $\psi(X \setminus E) = X \setminus E$. Verify that τ is a topology in X . Prove that, for each $E \subset X$, $\psi(E)$ is the closure of E in the space X with τ as topology.

2F. Let A and B be sets in a space X . Prove that

$$\partial(A \cup B) \subset \partial(A) \cup \partial(B),$$

and give an example to show that $\partial(A \cup B)$ may be different from $\partial(A) \cup \partial(B)$.

2G. Let E be a dense subset in a space X and U be an open subset of X . Prove that

$$U \subset \text{Cl}(E \cap U).$$

3. MAPS

Let us consider a function

$$f: X \rightarrow Y$$

of which both the domain X and the range Y are spaces. The function f is said to be *continuous at a point* p of X iff, for every neighborhood N of the point $f(p)$ in Y , there exists a neighborhood M of the point p in X such that $f(M) \subset N$. The function f is said to be *continuous* iff it is continuous at every point of X . Continuous functions will be called *mappings* or *maps*. For example, the constant functions from a space X into a space Y are continuous and will be called the *constant maps*. As another example, the identity function on a space X is continuous and will be called the *identity map* on X .

THEOREM 3.1. *If $f: X \rightarrow Y$ is a function from a space X into a space Y with a given basis and a given sub-basis of its topology, then the following statements are equivalent.*

- (i) *The function $f: X \rightarrow Y$ is a map.*
- (ii) *The inverse image $f^{-1}(U)$ of each open set U in Y is open in X .*
- (iii) *The inverse image $f^{-1}(V)$ of each basic open set V in Y is open in X .*
- (iv) *The inverse image $f^{-1}(W)$ of each sub-basic open set W in Y is open in X .*
- (v) *The inverse image $f^{-1}(F)$ of each closed set F in Y is closed in X .*
- (vi) *$f[\text{Cl}(A)] \subset \text{Cl}[f(A)]$ for each $A \subset X$.*
- (vii) *$f^{-1}[\text{Cl}(B)] \supset \text{Cl}[f^{-1}(B)]$ for each $B \subset Y$.*

Proof. (i) $\Rightarrow$ (ii). Let U be an arbitrary open set in Y . We are going to prove that $f^{-1}(U)$ is open in X . For this purpose, let p be any point in $f^{-1}(U)$. Then $f(p)$ is a point in U . Since U is open, it is a neighborhood of $f(p)$ in Y . Since f is continuous at the point p , there exists a neighborhood M of p in X such that $f(M) \subset U$. This implies that $M \subset f^{-1}(U)$. By (1.1), it follows that $f^{-1}(U)$ is an open set in X .

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (v). Let F be an arbitrary closed set in Y . We are going to prove that $f^{-1}(F)$ is closed in X . By (2.3), it suffices to show that $X \setminus f^{-1}(F)$ is open in X . For this purpose, let p be an arbitrary point in $X \setminus f^{-1}(F)$. Then $f(p) \in Y \setminus F$. Since F is closed in Y , $Y \setminus F$ is open in Y according to (2.3). By definition, there exists a finite number of sub-basic open sets of Y , say $W_1, \dots, W_n$, such that

$$f(p) \in W_1 \cap \dots \cap W_n \subset Y \setminus F.$$

This implies that

$$p \in f^{-1}(W_1) \cap \dots \cap f^{-1}(W_n) \subset X \setminus f^{-1}(F).$$

By (iv), $f^{-1}(W_i)$ is open for each $i = 1, \dots, n$. By (OS4), $f^{-1}(W_1) \cap \dots \cap f^{-1}(W_n)$ is open and hence a neighborhood of p in X . Then it follows from (1.1) that $X \setminus f^{-1}(F)$ is open.

(v) $\Rightarrow$ (vi). Let A be an arbitrary set in X . Then $\text{Cl}[f(A)]$ is a closed set in Y . By (v), $f^{-1}\{\text{Cl}[f(A)]\}$ is a closed set in X . Obviously, $A \subset f^{-1}\{\text{Cl}[f(A)]\}$. Since $\text{Cl}(A)$ is the smallest closed set in X containing A , we have $\text{Cl}(A) \subset f^{-1}\{\text{Cl}[f(A)]\}$. This implies

$$f[\text{Cl}(A)] \subset \text{Cl}[f(A)].$$

(vi) $\Rightarrow$ (vii). Let B be an arbitrary set in Y and let $A = f^{-1}(B)$. Then, by (vi), we have

$$f[\text{Cl}(A)] \subset \text{Cl}[f(A)] \subset \text{Cl}(B).$$

This implies that $\text{Cl}(A) \subset f^{-1}[\text{Cl}(B)]$. That is,

$$f^{-1}[\text{Cl}(B)] \supset \text{Cl}[f^{-1}(B)].$$

(vii) $\Rightarrow$ (i). Let p be an arbitrary point of X and N be any neighborhood of $f(p)$ in Y . Then, there exists an open set U of Y such that $f(p) \in U \subset N$. Consider the closed set $B = Y \setminus U$ in Y . According to (vii), we have

$$\text{Cl}[f^{-1}(B)] \subset f^{-1}[\text{Cl}(B)] = f^{-1}(B).$$

Hence $f^{-1}(B)$ is a closed set in X . This implies that the set $M = X \setminus f^{-1}(B)$ is an open neighborhood of p in X and that $f(M) \subset U \subset N$. Therefore, f is continuous at p . Since p is arbitrary, this completes the proof of (3.1). $\square$

A map $f: X \rightarrow Y$ from a space X into a space Y is said to be *open* iff the image $f(U)$ of every open set U in X is open in Y ; similarly,

$f: X \rightarrow Y$ is said to be *closed* iff the image $f(F)$ of every closed set F in X is closed in Y .

If a map $f: X \rightarrow Y$ is bijective, then its inverse function $f^{-1}: Y \rightarrow X$ is well-defined but not always continuous. For example, let X denote the space of all real numbers with the discrete topology and Y denote the real line. Then the identity function $i: X \rightarrow Y$ is a bijective map, but its inverse function $i^{-1}: Y \rightarrow X$ fails to be continuous. As to the continuity of the inverse function of a bijective map, the following proposition is obvious.

PROPOSITION 3.2. *For any bijective map $f: X \rightarrow Y$, the following statements are equivalent.*

- (a) *The inverse function $f^{-1}: Y \rightarrow X$ is continuous.*
- (b) *$f: X \rightarrow Y$ is open.*
- (c) *$f: X \rightarrow Y$ is closed.*

By a *homeomorphism*, or *topological map*, we mean a bijective map $f: X \rightarrow Y$ which satisfies the equivalent conditions (a), (b), (c) in (3.2). If a homeomorphism $h: X \rightarrow Y$ exists, then the two spaces X and Y are said to be *homeomorphic*, or *topologically equivalent*, and each space is said to be a *homeomorph* of the other. A property which when possessed by a space is also possessed by each of its homeomorphs is called a *topological property*. Formally, topology is the study of topological properties of spaces.

For any two composable functions

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

the relations in (2.3) of Chapter I imply the following proposition.

PROPOSITION 3.3. *If f and g are continuous, so is the composition $g \circ f$. If f and g are homeomorphisms, then f^{-1} , g^{-1} and $g \circ f$ are also homeomorphisms.*

Throughout the remainder of the section, we will consider a function

$$f: X \rightarrow Y$$

from a set X into a set Y .

Let τ be an arbitrary topology in X . Define a collection θ of subsets of Y as follows: a subset V of Y is in θ iff the inverse image $f^{-1}(V)$ is in τ . One easily verifies that the collection θ satisfies the axioms for open sets and hence is a topology in Y . In fact, θ is the largest topology in Y which makes f a map. This topology θ is said to be the *induced topology* in Y by the function f and the topology τ in X , and will be denoted by $f_*(\tau)$.

On the other hand, let η be an arbitrary topology in Y . Then it is obvious that the collection

$$\xi = \{f^{-1}(V) \mid V \in \eta\}$$

of subsets of X satisfies the axioms for open sets and hence is a topology in X . In fact, ξ is the smallest topology in X which makes f a map. This topology ξ is said to be the *induced topology* in X by the function f and the topology η in Y and will be denoted by $f^{-1}(\eta)$.

Examples of induced topologies of both kinds will be given in the following sections.

EXERCISES

3A. Consider a function $f: X \rightarrow Y$ from a space X into a space Y , a point $p \in X$, and a local basis of Y at the point $f(p)$. Prove that f is continuous at p iff, for every basic neighborhood N of $f(p)$ in Y , there exists a neighborhood M of p in X such that $f(M) \subset N$.

3B. Let E be any subset of a space X ; consider the characteristic function

$$\chi_E: X \rightarrow R$$

from the space X into the real line R as defined in (I, § 2, Example 4). Prove that χ_E is continuous at a point $p \in X$ iff p is not a boundary point of E .

3C. Let X denote the real line with the usual topology τ and let Y denote the set of all real numbers. Describe the open sets of the topology $f_*(\tau)$ in Y induced by the function

$$f: X \rightarrow Y$$

which is defined by taking $f(x) = |x|$, the absolute value of the real number x .

3D. Let E be a subset in a set X , R the real line with the usual topology τ , and $\chi_E: X \rightarrow R$ the characteristic function of E in X . Exhibit the open sets in the topology $\chi_E^{-1}(\tau)$ induced in the set X .

3E. Prove that every function from a discrete space into a space or from a space into an indiscrete space is always continuous.

4. SUBSPACES

Let E be a given subset of a space X with a topology τ . Then, the inclusion function

$$i: E \subset X$$

induces a topology $i^{-1}(\tau)$ in E as defined in the previous section. This topology $i^{-1}(\tau)$ in E will be called the *relative topology* in E with respect to τ . When topologized by the relative topology $i^{-1}(\tau)$, E will be called a *subspace* of the space X . It follows immediately from the definition of $i^{-1}(\tau)$ that a set $U \subset E$ is open in E iff there exists an open set V of X such that

$$U = i^{-1}(V) = E \cap V.$$

Similarly, a set $C \subset E$ is closed in E iff there exists a closed set D of X such that $C = E \cap D$.

EXAMPLES OF SUBSPACES. The subset

$$I = \{x \in R \mid 0 \leq x \leq 1\}$$

of the real line R furnished with the relative topology will be called the *unit interval*. The subspace Z of the real line R consisting of all integers, (positive, negative, or zero) is discrete.

Now, let E be an arbitrary subspace of a space X . It follows immediately from the definition that the inclusion function

$$i: E \subset X$$

is a map which will be called the *inclusion map*.

PROPOSITION 4.1. *A subspace E of a space X is open in X iff the inclusion map $i: E \subset X$ is open. Hence, every open subset of an open subspace is open.*

Proof: Necessity. Assume that E is open in X . We are going to prove that $i: E \subset X$ is an open map. For this purpose, let U be any open set of the subspace E . By definition, there exists an open set V of the space X such that

$$i(U) = U = E \cap V.$$

Since E and V are open in X , so is U . This implies that i is an open map.

Sufficiency. Assume that i is an open map. Since E is open in E and i is an open map, it follows that $E = i(E)$ is open in X . ||

Similarly, one can prove the following proposition.

PROPOSITION 4.2. *A subspace E of a space X is closed in X iff the inclusion map $i: E \subset X$ is closed. Hence, every closed subset of a closed subspace is closed.*

Next, let us consider two spaces X and Y together with a subspace E of X and the inclusion map $i: E \subset X$.

For an arbitrary map $f: X \rightarrow Y$, the composed map

$$g = f \circ i: E \rightarrow Y$$

will be called the *restriction* of the map f on the subspace E of X , denoted by

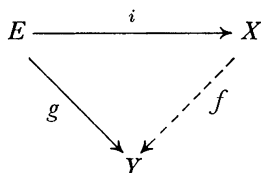
$$g = f|E.$$

For every point $p \in E$, we have

$$g(p) = f[i(p)] = f(p);$$

hence g and f are essentially the same map except that the domain of g has been restricted to the subspace E .

By an *extension* of a map $g: E \rightarrow Y$ over the space X , we mean a map $f: X \rightarrow Y$ such that $f|E = g$. The *extension problem* of a given map $g: E \rightarrow Y$ over the space X is to determine whether or not g has an extension over X and to find an extension of g over X if such exists. In other words, the extension problem of a given map $g: E \rightarrow Y$ over X is to find a map $f: X \rightarrow Y$ which makes the triangle



commutative, that is, $f \circ i = g$ where i stands for the inclusion map. As we shall see later, the answer to the extension problem is not always affirmative.

Consider the special case where $Y = E$ and $g: E \rightarrow Y$ is the identity map on E . Then an extension of g over the space X will be a map $r: X \rightarrow E$ such that $r|E$ is the identity map on E . Such a map $r: X \rightarrow E$ will be called a *retraction* of X onto E . If a retraction of X onto E exists, the subspace E of X is said to be a *retract* of the space X .

EXAMPLES OF RETRACTS. The space X is a retract of itself. Every singleton subspace of the space X is a retract of X . Finally, the unit interval I is a retract of the real line R with a retraction $r: R \rightarrow I$ defined by

$$r(t) = \begin{cases} 0, & (\text{if } t \leq 0), \\ t, & (\text{if } 0 \leq t \leq 1), \\ 1, & (\text{if } t \geq 1). \end{cases}$$

This last example shows that frequently a function is constructed

by prescribing it on pieces of its domain. In the next few propositions, we will give sufficient conditions for the continuity of functions so constructed.

By a *covering*, or *cover*, of a space X , we mean a collection γ of subspaces of the space X such that the union of all members of γ is the whole space X . If every member of γ is open in X , γ is said to be an *open cover* of X ; if every member of γ is closed in X , γ is said to be a *closed cover* of X .

Let γ be a given cover of a space X . For an arbitrary map $g: X \rightarrow Y$ from the space X into a space Y , the restrictions

$$g_A = g \mid A: A \rightarrow Y, \quad (A \in \gamma),$$

are well-defined maps and satisfy the relations

$$g_A \mid A \cap B = g_B \mid A \cap B, \quad (A, B \in \gamma).$$

Our problem is to study the inverse of this process described as follows.

Let us assume that, for each member $A \in \gamma$, there is given a map $f_A: A \rightarrow Y$ such that

$$f_A \mid A \cap B = f_B \mid A \cap B$$

for each pair of members A and B of γ . Then we may define a function $f: X \rightarrow Y$ by taking

$$f(x) = f_A(x), \quad (\text{if } x \in A \in \gamma).$$

We are concerned with the problem of whether or not f is continuous.

PROPOSITION 4.3. *If γ is a finite closed cover of X , then the combined function f is continuous.*

Proof. Let F be any closed set in Y . For each member $A \in \gamma$, it follows from the continuity of f_A that $f_A^{-1}(F)$ is a closed set of A according to (v) in (3.1). Since A is closed in X , it follows from (4.2) that $f_A^{-1}(F)$ is also closed in X . Since γ is finite,

$$f^{-1}(F) = \bigcup_{A \in \gamma} f_A^{-1}(F)$$

is a closed set of X according to (CS3) in (2.4). Hence f is continuous. ||

PROPOSITION 4.4. *If p is an interior point of some member A of γ , then the combined function f is continuous at p .*

Proof. Let N be an arbitrary neighborhood of the point $f(p) = f_A(p)$ in Y . It follows from the continuity of f_A that there exists a neighborhood M of p in A such that $f_A(M) \subset N$. By the definition of a neighborhood, there exists an open set U of A with $p \in U \subset M$. Let

$$V = U \cap \text{Int}(A).$$

Since $\text{Int}(A)$ is open in X and V is open in $\text{Int}(A)$, it follows from (4.1) that V is open in X . Since $p \in V \subset M$, we have

$$f(V) = f_A(V) \subset N.$$

Hence f is continuous at p . $\square$

COROLLARY 4.5. *If γ is an open cover of X , then the combined function f is continuous.*

Next, we will generalize the notion of the restrictions of a map as follows. Let

$$f: X \rightarrow Y$$

be a given map from a space X into a space Y and consider subspaces

$$A \subset X, \quad B \subset Y$$

such that $f(A) \subset B$. Define a function

$$g: A \rightarrow B$$

by assigning $g(x) = f(x)$ for every x in A . One can easily verify that g is a map which will be referred to as the *map from A into B defined by f* .

Let $i: A \subset X$ and $j: B \subset Y$ denote the inclusion maps. Then the rectangle

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \downarrow i & & \downarrow j \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative, that is, $f \circ i = j \circ g$. In case $B = Y$, then j becomes the identity map and $g = f \circ i = f|A$ reduces to the restriction of f on A .

Finally, we will introduce the important notion of an imbedding as follows.

By an *imbedding* of a space X into a space Y , we mean an injective map $f: X \rightarrow Y$ which defines a homeomorphism from X onto $f(X)$. For example, every inclusion map is an imbedding.

EXERCISES

4A. The open interval (a, b) , $a < b$, as a subspace of the real line R , is defined by

$$(a, b) = \{t \in R \mid a < t < b\}.$$

Prove that (a, b) and R are homeomorphic by studying the function $h: (a, b) \rightarrow R$ defined by

$$h(t) = \begin{cases} \frac{t-c}{t-a}, & (\text{if } a < t \leq c), \\ \frac{t-c}{b-t}, & (\text{if } c \leq t < b), \end{cases}$$

where c denotes the arithmetic mean of a and b .

- 4B. Let A , B , and C be three subspaces of a space X such that $C \subset A \cap B$. Prove that C is open in the subspace $A \cup B$ if it is open in both A and B , and prove that C is closed in the subspace $A \cup B$ if it is closed in both A and B .
- 4C. Let A and B be two sets in a space X such that $B \subset A$. Prove:
- (a) $\text{Int}(B)$ is contained in the interior of B with respect to the subspace A .
 - (b) $\text{Cl}(B) \cap A$ is the closure of B with respect to the subspace A .
 - (c) $\partial(B) \cap A$ contains the boundary of B with respect to subspace A .
- Can the words “is contained in” and “contains” in (a) and (c) be replaced by “is identical with”?
- 4D. Let A and B be two sets in a space X such that $A \cup B = X$. Consider two maps $f, g: X \rightarrow Y$ and a given point y_0 in the space Y such that $f(\text{Cl } B) = y_0 = g(\text{Cl } A)$. Define a function $h: X \rightarrow Y$ by taking

$$h(x) = \begin{cases} f(x), & (\text{if } x \in A), \\ g(x), & (\text{if } x \in B). \end{cases}$$

Verify that h is well-defined and continuous.

- 4E. PROVE LINDELÖF'S THEOREM: *Every open cover of a space satisfying the second axiom of countability has a countable subcover.* A space is said to be a *Lindelöf space* iff each open cover of the space has a countable subcover.

5. SUM AND PRODUCT OF SPACES

Consider a given indexed family

$$\mathcal{F} = \{X_\mu \mid \mu \in M\}$$

of sets. If each of the sets X_μ is a space, $\mathcal{F}$ is called an *indexed family of spaces*. Furthermore, if $\mu \neq \nu$ implies that

$$X_\mu \cap X_\nu = \emptyset,$$

then the family $\mathcal{F}$ is said to be *disjoint*.

To introduce the notion of the sum of spaces, let us consider an arbitrarily given indexed family

$$\mathcal{F} = \{X_\mu \mid \mu \in M\}$$

of spaces and denote by X the union of all sets X_μ , that is,

$$X = \bigcup_{\mu \in M} X_\mu.$$

Define a collection τ of subsets of X as follows: A subset U of X is in τ iff $U \cap X_\mu$ is an open set of the space X_μ for every $\mu \in M$. It is obvious that τ satisfies the axioms (OS1)–(OS4) for open sets and hence is a topology in X . When topologized by τ , the space X will be referred to as the *topological sum*, or *sum*, of the family $\mathcal{F}$ of spaces and denoted by

$$X = \sum_{\mu \in M} X_\mu.$$

In applications, the given family $\mathcal{F}$ of spaces is usually disjoint.

Since $X_\mu \subset X$ for each $\mu \in M$, the inclusion functions

$$i_\mu: X_\mu \subset X, \quad (\mu \in M)$$

are well-defined.

PROPOSITION 5.1. *For each $\mu \in M$, the inclusion function*

$$i_\mu: X_\mu \subset X$$

is continuous. If the indexed family $\mathcal{F}$ of spaces is disjoint, then each i_μ is both open and closed.

Proof. To prove the continuity of i_μ , let U be an arbitrary open set of X . Then $U \cap X_\mu$ is an open set of X_μ according to the definition of the topology in X . Since

$$i_\mu^{-1}(U) = U \cap X_\mu,$$

it follows from (ii) of (3.1) that i_μ is continuous.

Next, assume that $\mathcal{F}$ is disjoint. Let U be any open set in X_μ . Since $\mathcal{F}$ is disjoint, we have $U \cap X_\mu = U$ and $U \cap X_\nu = \emptyset$ for each $\nu \neq \mu$. Hence U is also an open set of X . Since $i_\mu(U) = U$, this proves that i_μ is open. On the other hand, let F be any closed set of X_μ and consider the set $V = X \setminus F$. Since $V \cap X_\mu = X_\mu \setminus F$ and $V \cap X_\nu = X_\nu$ for each $\nu \neq \mu$, it follows that V is an open set of X . This implies that $i_\mu(F) = F$ is a closed set of X and hence i_μ is closed. ||

COROLLARY 5.2. *If the indexed family $\mathcal{F}$ is disjoint, then each i_μ is an imbedding of X_μ as an open and closed subspace of X .*

Next, to introduce the notion of products of spaces, let

$$\mathcal{F} = \{X_\mu \mid \mu \in M\}$$

be an arbitrarily given indexed family of spaces and consider the Cartesian product

$$\Phi = \prod_{\mu \in M} X_\mu$$

defined in (I, § 3). Then Φ consists of all functions $f: M \rightarrow X$ of M into the union X of all sets X_μ in $\mathcal{F}$ such that $f(\mu) \in X_\mu$ for every $\mu \in M$.

We are going to define a topology τ in Φ as follows. For an arbitrary $\mu \in M$, and any open set U_μ of X_μ , the set

$$U_\mu^* = \{f \in \Phi \mid f(\mu) \in U_\mu\}$$

will be called a *sub-basic open set* of Φ defined by $U_\mu \subset X_\mu$. Let σ denote the collection of all sub-basic open sets of Φ . Then we define topology τ to be the smallest topology in Φ containing the collection σ . This topology τ will be called the *product topology* in Φ . When topologized with τ , Φ is said to be the *topological product*, or *product*, of the family $\mathcal{F}$ of spaces, also denoted by

$$\Phi = \prod_{\mu \in M} X_\mu.$$

Since σ is obviously a cover of Φ and since τ is the smallest topology in Φ containing σ , it follows from Exercise 1G that σ constitutes a sub-basis of τ and that the collection β of all finite intersections of sub-basic open sets of Φ forms a basis of τ .

EXAMPLES OF PRODUCT SPACES. If each member X_μ of the family $\mathcal{F}$ is equal to a given space X , then the topological product Φ of the family $\mathcal{F}$ will be referred to as the *Mth topological power* of the space X denoted by

$$\Phi = X^M.$$

In particular, if M consists of the first n natural numbers, $1, \dots, n$, Φ is called the *nth topological power* of the space X ; in symbols,

$$\Phi = X^n.$$

The *nth topological power* R^n of the real line R is called the *n-dimensional Euclidean space*. Hence, the points of R^n are the ordered *n-tuples* $(x_1, \dots, x_n)$ of real numbers. Similarly, the *nth topological power* I^n of the unit interval I is called the *unit n-cube*. The *Nth topological power* I^N of the unit interval I , where N denotes the set of all natural numbers, is called the *Hilbert cube*.

Now, let Φ denote the topological product of an indexed family

$$\mathcal{F} = \{X_\mu \mid \mu \in M\}$$

of spaces. If $X_\mu = \square$ for some $\mu \in M$, then we have $\Phi = \square$. Hereafter, we assume that $X_\mu \neq \square$ for all $\mu \in M$.

For each $\mu \in M$, consider the projection

$$p_\mu : \Phi \rightarrow X_\mu$$

defined by $p_\mu(f) = f(\mu)$ for every $f \in \Phi$ in (I, § 3). Then, p_μ is surjective for every $\mu \in M$. We will call p_μ the *projection* of the topological product Φ onto its μ -th coordinate space X_μ .

PROPOSITION 5.3. *The projection $p_\mu : \Phi \rightarrow X_\mu$ is an open map from Φ onto X_μ for each $\mu \in M$.*

Proof. To prove that p_μ is continuous, let U_μ be any open set in X_μ . Then

$$p_\mu^{-1}(U_\mu) = \{f \in \Phi \mid f(\mu) \in U_\mu\} = U_\mu^*$$

is a sub-basic open set of Φ . Hence p_μ is continuous.

Next, to prove that p_μ is open, let U be an arbitrary open set of Φ . We are going to prove that $p_\mu(U)$ is an open set of X_μ . Let $a \in p_\mu(U)$. Then there exists a point $f \in U$ such that $f(\mu) = a$. Since σ is sub-basis of the product topology, there exist sub-basic open sets $U_{\mu_i}^*$, ($i = 1, \dots, n$), such that

$$f \in V = U_{\mu_1}^* \cap \dots \cap U_{\mu_n}^* \subset U.$$

This implies that

$$a = p_\mu(f) \in p_\mu(V) \subset p_\mu(U).$$

It remains to prove that $p_\mu(V)$ is an open set of X_μ . This follows from the fact that $p_\mu(V) = U_{\mu_i}$ if $\mu = \mu_i$ for some i and that $p_\mu(V) = X_\mu$ if μ is different from μ_i for all $i = 1, \dots, n$. ||

There is a useful characterization of continuity of a function from a space into a topological product.

PROPOSITION 5.4. *A function $f: W \rightarrow \Phi$ from a space W into the topological product Φ is continuous iff, for each $\mu \in M$, the composition $p_\mu \circ f$ is continuous.*

Proof. Since the necessity of the condition is obvious, it remains to prove the sufficiency. Assume that $p_\mu \circ f$ is continuous for each $\mu \in M$ and consider an arbitrary sub-basic open set U_μ^* of Φ . Since $p_\mu \circ f$ is continuous, the set $(p_\mu \circ f)^{-1}(U_\mu)$ is an open set of W . Since

$$(p_\mu \circ f)^{-1}(U_\mu) = f^{-1}[p_\mu^{-1}(U_\mu)] = f^{-1}(U_\mu^*),$$

$f^{-1}(U_\mu^*)$ is an open set of W . Hence, it follows from (iv) of (3.1) that f is continuous. ||

For the special case of topological powers X^M , we have the following

PROPOSITION 5.5. *The diagonal injection (defined in (I, § 3))*

$$d: X \rightarrow X^M$$

of a space X into the topological power X^M is an imbedding, called the diagonal imbedding.

Proof. To prove the continuity of d , let U_μ^* denote an arbitrary sub-basic open set of X^M defined by an open set U_μ of $X_\mu = X$. Then, the inverse image

$$d^{-1}(U_\mu^*) = U_\mu$$

is open in X . By (iv) of (3.1), d is a map.

Let $d(X)$ denote the image of d in X^M with relative topology. Then, d defines a map

$$\delta: X \rightarrow d(X).$$

We will prove that δ is a homeomorphism.

Since δ is both bijective and continuous, it remains to prove that δ is open. For this purpose, let U be an arbitrary open set of X . Pick an element $\mu \in M$ and let

$$U_\mu = U \subset X = X_\mu.$$

Then U_μ^* is a sub-basic open set of X^M . Since

$$d(U) = U_\mu^* \cap d(X),$$

$d(U)$ is an open set of $d(X)$. ||

The image $d(X)$ in X^M of the diagonal imbedding d is called the *diagonal* of the topological power X^M . By means of the diagonal imbedding d , the given space X can be identified with the diagonal $d(X)$ of X^M and considered as a subspace of X^M .

PROPOSITION 5.6. *The diagonal $d(X)$ of a topological power X^M is a retract of X^M .*

Proof. Pick an element $\mu \in M$ and consider the composed map

$$r = \delta \circ p_\mu: X^M \rightarrow d(X)$$

of the projection $p_\mu: X^M \rightarrow X$ and the homeomorphism $\delta: X \rightarrow d(X)$ defined by the diagonal imbedding $d: X \rightarrow X^M$. Since r is obviously a retraction of X^M onto $d(X)$, $d(X)$ is a retract of X^M . ||

Finally, let us consider an arbitrarily given indexed family

$$\mathcal{H} = \{h_\mu : X_\mu \rightarrow Y_\mu \mid \mu \in M\}$$

of maps. Then the Cartesian product

$$H = \prod_{\mu \in M} h_\mu : \Phi \rightarrow \Psi,$$

as defined in (I, § 3), is a function of the topological product Φ of the spaces X_μ into the topological product Ψ of the spaces Y_μ . Since H is a function from a space Φ into a space Ψ , we are naturally interested in the continuity of H .

PROPOSITION 5.7. *The Cartesian product of any indexed family of continuous functions is continuous with respect to the product topologies.*

Proof. We will prove the continuity of $H : \Phi \rightarrow \Psi$. For this purpose, let V_μ^* denote an arbitrary sub-basic open set of Ψ defined by an open set V_μ in Y_μ with some $\mu \in M$. Since h_μ is continuous, the inverse image

$$U_\mu = h_\mu^{-1}(V_\mu) \subset X_\mu$$

is an open set in X_μ and hence defines a sub-basic open set U_μ^* of Φ . Since obviously

$$U_\mu^* = H^{-1}(V_\mu^*),$$

it follows from (iv) of (3.1) that H is continuous. ||

The following corollary is a direct consequence of (5.5) and (5.7).

COROLLARY 5.8. *The restricted Cartesian product of an arbitrarily given indexed family*

$$\mathcal{H} = \{h_\mu : X \rightarrow Y_\mu \mid \mu \in M\}$$

of continuous functions defined on the same space X is continuous.

EXERCISES

5A. Let $\mathcal{F} = \{X_\mu \mid \mu \in M\}$ be an indexed family of spaces, where X_μ denotes the space obtained by topologizing a set X by a topology τ_μ . Prove that the topological sum

$$\sum_{\mu \in M} X_\mu$$

is the space obtained by topologizing X with the topology

$$\tau = \inf_{\mu \in M} \tau_\mu = \bigcap_{\mu \in M} \tau_\mu.$$

- 5B. Consider the topological product $\Phi = X_1 \times X_2$ of two spaces X_1 and X_2 . Let $a_1 \in X_1$ and $a_2 \in X_2$. Define two functions

$$i_1: X_1 \rightarrow \Phi, \quad i_2: X_2 \rightarrow \Phi$$

by taking $i_1(x_1) = (x_1, a_2) \in \Phi$ for each $x_1 \in X_1$ and $i_2(x_2) = (a_1, x_2) \in \Phi$ for each $x_2 \in X_2$. Prove that i_1 and i_2 are imbeddings and $p_1 \circ i_1$ and $p_2 \circ i_2$ are the identity maps on X_1 and X_2 respectively. Generalize these facts to arbitrary topological products.

- 5C. In the preceding exercise, let $E_1 \subset X_1$ and $E_2 \subset X_2$. Denote by $E_1 \times E_2$ the subset of the topological product $X_1 \times X_2$ which consists of the points (x_1, x_2) such that $x_1 \in E_1$ and $x_2 \in E_2$. Prove:
- (a) $\text{Cl}(E_1 \times E_2) = (\text{Cl } E_1) \times (\text{Cl } E_2)$;
 - (b) $\text{Int}(E_1 \times E_2) = (\text{Int } E_1) \times (\text{Int } E_2)$;
 - (c) $\partial(E_1 \times E_2) = (\partial E_1 \times \text{Cl } E_2) \cup (\text{Cl } E_1 \times \partial E_2)$.

- 5D. Let $\mathcal{F} = \{X_\mu \mid \mu \in M\}$ be any indexed family of spaces. For each $\mu \in M$, we will also denote by μ the singleton space $\{\mu\}$. Let X'_μ denote the topological product $\mu \times X_\mu$. Show the family $\mathcal{F}' = \{X'_\mu \mid \mu \in M\}$ is disjoint. The topological sum of the family $\mathcal{F}'$ is called the *disjoint sum* of $\mathcal{F}$.

- 5E. Let E^n denote the *unit open n -cell* in the n -dimensional Euclidean space R^n which consists of all points $(x_1, \dots, x_n)$ of R^n with $x_1^2 + \dots + x_n^2 < 1$. Prove that E^n and R^n are homeomorphic.
- 5F. Let S^n denote the *unit n -sphere* in the $(n+1)$ -dimensional Euclidean space R^{n+1} which consists of all points $(x_0, x_1, \dots, x_n)$ of R^{n+1} satisfying the equation $x_0^2 + x_1^2 + \dots + x_n^2 = 1$. Let p be a point in S^n . Prove that $S^n \setminus p$ is homeomorphic to the n -dimensional Euclidean space R^n .

- 5G. Let Δ^n denote the *unit n -simplex* in the Euclidean space R^{n+1} which consists of all points $(x_0, x_1, \dots, x_n)$ in R^{n+1} such that $x_i \geq 0$ for each $i = 0, 1, \dots, n$ and that $x_0 + x_1 + \dots + x_n = 1$. Prove that Δ^n is homeomorphic to the unit n -cube I^n and to the *closed unit n -cell* $\text{Cl}(E^n)$ in R^n .

- 5H. Prove that the subspace

$$T = [\text{Cl}(E^n) \times \{0\}] \cup [S^{n-1} \times I]$$

of the topological product $P = \text{Cl}(E^n) \times I$ is a retract of P .

- 5I. Show that the projection $p: R^2 \rightarrow R$ of the Euclidean plane R^2 onto its first coordinate axis is not closed.
- 5J. Prove the assertion: If each space of a countable family satisfies the second axiom of countability, so does the topological product

of the family. In particular, the spaces R^n , R^N satisfy the second axiom of countability.

6. IDENTIFICATION AND QUOTIENT SPACES

LEMMA 6.1. *For any map $f: X \rightarrow Y$, the following two statements are equivalent:*

- (a) *A set $E \subset Y$ is open in Y iff $f^{-1}(E)$ is open in X .*
- (b) *A set $E \subset Y$ is closed in Y iff $f^{-1}(E)$ is closed in X .*

Proof. The equivalence (a) $\Leftrightarrow$ (b) is an immediate consequence of the relation

$$f^{-1}(Y \setminus E) = X \setminus f^{-1}(E). \quad ||$$

A surjective map $f: X \rightarrow Y$ is said to be an *identification* iff it satisfies the equivalent conditions (a) and (b) in (6.1).

THEOREM 6.2. *If $f: X \rightarrow Y$ is a surjective open or closed map, then f is an identification.*

Proof. First, let us assume that f is open. Let E be a set in Y such that $f^{-1}(E)$ is open in X . Since f is surjective, we have

$$f[f^{-1}(E)] = E.$$

Since $f^{-1}(E)$ is open in X and f is open, it follows that E is open in Y . This proves that f satisfies the condition (a) in (6.1) and hence is an identification.

Similarly, one can prove that f is an identification if f is closed. $||$

In particular, every homeomorphism is an identification.

THEOREM 6.3. *If $f: X \rightarrow Y$ is an identification and $g: Y \rightarrow Z$ is a function of Y into a space Z , then a necessary and sufficient condition for the continuity of g is that of the composition $g \circ f$.*

Proof. The necessity of the condition is obvious since the composition of two maps is a map.

To prove the sufficiency, let us assume that the composition

$$h = g \circ f: X \rightarrow Z$$

is continuous. Let W be an arbitrary open set in Z and let $V = g^{-1}(W)$ and $U = f^{-1}(V)$. Since $h = g \circ f$, we have

$$h^{-1}(W) = f^{-1}[g^{-1}(W)] = U.$$

Since W is open and h is continuous, it follows that U is open in X . By the definition of an identification, this implies that V is open in Y . Hence, g is continuous. ||

Now, let $f: X \rightarrow Y$ be a surjective function from a space onto a set Y . According to § 3, this function f induces a topology in Y as follows: a set $V \subset Y$ is open iff $f^{-1}(V)$ is open in X . This topology is usually called the *identification topology* in Y with respect to f . If Y is topologized by this topology, then f clearly becomes an identification.

Next, let X be any given space. By a *decomposition*, or *partition*, of X , we mean a disjoint family Q of nonempty subsets of X covering X ; in other words, Q is a family of nonempty subsets of X such that each point of X belongs to one and only one member of Q . The function

$$p: X \rightarrow Q$$

which assigns to each point $x \in X$ the unique member $p(x)$ of Q containing x is called the *natural projection* of X onto Q . We give Q the identification topology with respect to p and obtain a space Q , which is usually called a *decomposition space* of X .

In particular, if, in a space X , there is given an equivalence relation, i.e. a reflexive, symmetric, and transitive relation $\sim$ defined between pairs of points of X , then the points of X are divided into disjoint subsets called the *equivalence classes*. Let Q denote the set of all equivalence classes. Then Q is a disjoint family of nonempty subsets of X which covers X and hence is a decomposition of X . The decomposition space Q so obtained is called the *quotient space* of X over the equivalence relation $\sim$, in symbols,

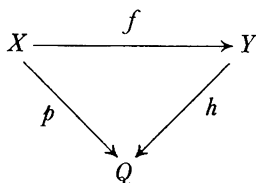
$$Q = X/\sim.$$

Conversely, let Q be any decomposition of X . Then we can define a relation $\sim$ in X as follows: For any two points a and b in X , $a \sim b$ iff there exists a member of Q which contains both a and b . One can easily verify that this relation $\sim$ is an equivalence relation and that Q consists of precisely all equivalence classes. Hence, every decomposition space of X is a quotient space over an equivalence relation.

Now, let us go back to study an arbitrary identification $f: X \rightarrow Y$. Define a relation $\sim$ in X as follows: For any two points a and b in X , $a \sim b$ iff $f(a) = f(b)$. It is easily verified that $\sim$ is an equivalence relation in X and hence defines a quotient space $Q = X/\sim$. The members of Q are precisely the inverse images $f^{-1}(y)$ for the points y of Y . The assignment $y \rightarrow f^{-1}(y)$ defines a bijective function

$$h: Y \rightarrow Q.$$

Since both $f: X \rightarrow Y$ and the natural projection $p: X \rightarrow Q$ are identifications and the triangle



is commutative, it follows from (6.3) that h is a homeomorphism. This shows that every identification is essentially the natural projection of a space onto its quotient space.

EXAMPLES OF QUOTIENT SPACES.

(1) *The real projective space P^n .* Consider the $(n+1)$ -dimensional Euclidean space R^{n+1} and let

$$X = R^{n+1} \setminus O, \quad O = (0, \dots, 0).$$

Introduce an equivalence relation $\sim$ in X as follows: For any two points $a = (a_0, a_1, \dots, a_n)$ and $b = (b_0, b_1, \dots, b_n)$ in X , $a \sim b$ iff there exists a nonzero real number λ such that

$$b_i = \lambda a_i, \quad (i = 0, 1, \dots, n).$$

The quotient space $X/\sim$ thus obtained is called the n -dimensional real projective space P^n . Hence, the points of P^n are the straight lines in R^{n+1} passing through the origin O .

Let $p: X \rightarrow P^n$ denote the natural projection. Since the unit n -sphere S^n is a subspace of X , p defines a restriction

$$\pi = p|S^n: S^n \rightarrow P^n.$$

Since every point $x = (x_0, x_1, \dots, x_n)$ in X can be normalized by multiplying X with the nonzero real number

$$\lambda = \frac{1}{|x|}, \quad |x|^2 = x_0^2 + x_1^2 + \dots + x_n^2,$$

the map π is surjective. One can easily see that π is an identification.

If $a = (a_0, a_1, \dots, a_n)$ and $b = (b_0, b_1, \dots, b_n)$ be any two points of S^n , then $\pi(a) = \pi(b)$ iff $a_i + b_i = 0$ for every $i = 0, 1, \dots, n$; in other words, a and b are antipodal points. Hence P^n can be obtained from S^n by identifying the antipodal points.

(2) *Collapsing a subset to a point.* Consider a given space X and a

given nonempty subset A in X . The individual points of $X \setminus A$ together with the subset A form a decomposition of the space X . The quotient space Q thus defined is called the *space obtained from X by collapsing the subset A to a point*. We will give three special cases as examples.

First, let $X = \text{Cl}(E^n)$ denote the closed unit n -cell in R^n and $A = S^{n-1}$ the unit $(n-1)$ -sphere as defined in Exercises 5F and 5G. Then it is not difficult to show that the quotient space obtained from $\text{Cl}(E^n)$ by collapsing its boundary S^{n-1} to a point is homeomorphic to the unit n -sphere S^n in R^{n+1} .

Second, let $X = E \times I$ denote the topological product of a space E and the closed unit interval I , and let $A = E \times \{1\}$. Then the quotient space obtained from $E \times I$ by collapsing its top $E \times \{1\}$ to a point is called the *cone* over the space E , denoted by $\text{Con}(E)$. The point $v = E \times \{1\}$ of $\text{Con}(E)$ is called the *vertex*. The given space E can be considered as a subspace of $\text{Con}(E)$ by means of the imbedding $i: E \rightarrow \text{Con}(E)$ defined by $i(e) = p(e, 0)$ for each $e \in E$, where p denotes the natural projection of $E \times I$ onto $\text{Con}(E)$.

Third, let $X = U + V$ denote the topological sum of two disjoint nonempty spaces U and V , and let A denote the subspace of X which consists of a given point $u_0 \in U$ and a given point $v_0 \in V$. Then the quotient space obtained from $U + V$ by collapsing $\{u_0, v_0\}$ to a point is called the *one-point union* of the spaces U and V with base points $u_0 \in U$ and $v_0 \in V$, denoted by $U \vee V$. The given spaces U and V can be considered as subspaces of $U \vee V$ in the obvious way, and $U \vee V$ can be considered as a subspace of the topological product $U \times V$ by means of the imbedding $i: U \vee V \rightarrow U \times V$ defined by $i(u) = (u, v_0)$ for each $u \in U$ and $i(v) = (u_0, v)$ for each $v \in V$.

(3) *Mapping cylinder and mapping cone.* Let $f: X \rightarrow Y$ be any given map and consider the topological sum

$$E = (X \times I) + Y$$

of the topological product $X \times I$ and Y , where I denotes the unit interval. Then the individual points

$$\{(x, t) \in X \times I \mid x \in X, 0 \leq t < 1\}$$

together with the subsets

$$\{[f^{-1}(y) \times 1] \cup y \mid y \in Y\}$$

of E form a decomposition of E . The quotient space thus obtained is called the *mapping cylinder* of the map f , denoted by $\text{Cyl}(f)$. Both X

and Y can be considered as subspaces of $\text{Cyl}(f)$ by means of the imbeddings

$$i: X \rightarrow \text{Cyl}(f), \quad j: Y \rightarrow \text{Cyl}(f).$$

defined by $i(x) = p(x, 0)$ for each $x \in X$ and $j(y) = p(y)$ for each $y \in Y$, where p denotes the natural projection of E onto $\text{Cyl}(f)$. The quotient space obtained from the mapping cylinder $\text{Cyl}(f)$ by collapsing its subspace X to a point v is called the *mapping cone* of f , denoted by $\text{Con}(f)$. The point v is called the *vertex* of $\text{Con}(f)$. The space Y can be considered as a subspace of $\text{Con}(f)$ in the obvious way.

EXERCISES

6A. Show that the natural projection p of a topological product $X \times Y$ onto X is an identification. Hence, by Exercise 5 I, an identification may fail to be closed. Construct an example showing that an identification may fail to be open.

6B. Let Q be a quotient space of a space X with natural projection

$$p: X \rightarrow Q.$$

For each subset A of X , let

$$R(A) = \{x \in X \mid x \sim \text{some } a \in A\}.$$

$$S(A) = \{a \in A \mid x \sim a \text{ implies } x \in A\}.$$

Then prove that the following statements are equivalent:

- (a) p is an open map.
- (b) $R(A)$ is open for every open $A \subset X$.
- (c) $S(A)$ is closed for every closed $A \subset X$.

Show that the statements (a), (b), (c) remain equivalent if we interchange the words "open" and "closed."

6C. Consider the closed unit n -cell $\text{Cl}(E^n)$ in R^n . Show that the quotient space obtained from $\text{Cl}(E^n)$ by identifying the antipodal points on the boundary S^{n-1} is homeomorphic to the n -dimensional real projective space P^n .

6D. The points of the unit circle S^1 in R^2 can be considered as the complex numbers z with $|z| = 1$. Define a map $f: S^1 \rightarrow S^1$ by taking $f(z) = z^2$ for every $z \in S^1$. Prove that the mapping cone $\text{Con}(f)$ is homeomorphic to the real projective plane P^2 .

6E. Consider the unit square $I^2 = I \times I$. The quotient space obtained from I^2 by identifying $(0, t)$ with $(1, 1 - t)$ for every $t \in I$ is called the *Möbius strip*. Show by making a physical model that the

Möbius strip M can be imbedded in the 3-dimensional Euclidean space R^3 . The subspace

$$E = p[(I \times 0) \cup (I \times 1)]$$

of M , where p denotes the natural projection from I^2 onto M , is called the *edge* of M . Prove that E is homeomorphic to S^1 . Let $h: S^1 \rightarrow M$ be an imbedding which maps S^1 homeomorphically onto E . Show that the mapping cone $\text{Con}(h)$ is homeomorphic to the real projective plane P^2 . Then show that P^2 can be imbedded in the 4-dimensional Euclidean space R^4 .

6F. The quotient space obtained from I^2 by identifying $(0, t)$ with $(1, t)$, and $(t, 0)$ with $(t, 1)$, for every $t \in I$, is called the *torus* T^2 . Prove that T^2 is homeomorphic to the topological product $S^1 \times S^1$.

6G. The quotient space obtained from I^2 by identifying $(0, t)$ with $(1, 1 - t)$, and $(t, 0)$ with $(t, 1)$, for every $t \in I$, is called the *Klein bottle* K^2 . Prove that K^2 is homeomorphic to the quotient space obtained from the topological sum of two Möbius strips by identifying the corresponding points on their edges.

6H. The quotient space obtained from the cone $\text{Con}(E)$ over a space E by collapsing E to a point u is called the *suspension* of the space E . The point u and the vertex v of $\text{Con}(E)$ are called the *poles* of the suspension. Prove that the suspension of the unit n -sphere S^n is homeomorphic to S^{n+1} .

7. HOMOTOPY AND ISOTOPY

Let X and Y be given spaces and consider a family

$$\{h_t: X \rightarrow Y \mid t \in I\}$$

of maps indexed by the real numbers of the unit interval $I = [0, 1]$. The family $\{h_t\}$ is said to be *continuous* iff the function

$$H: X \times I \rightarrow Y,$$

defined by $H(x, t) = h_t(x)$ for each $x \in X$ and each $t \in I$, is continuous. A continuous family $\{h_t\}$ of maps, or equivalently the map H , is usually called a *homotopy*. Furthermore, if h_t is an imbedding for every $t \in I$, the continuous family $\{H_t\}$ is called an *isotopy*.

Now, let $f, g: X \rightarrow Y$ be two given maps. If there exists a homotopy $\{h_t: X \rightarrow Y \mid t \in I\}$ such that $h_0 = f$ and $h_1 = g$, f and g are said to be

homotopic, in symbols, $f \simeq g$. Intuitively, $f, g : X \rightarrow Y$ are homotopic iff f can be continuously changed into g .

PROPOSITION 7.1. *Any two maps*

$$f, g : X \rightarrow R^n$$

from a space X into the Euclidean space R^n are homotopic.

Proof. We will make use of the fact that R^n is a vector space over the real numbers. Define a function

$$H : X \times I \rightarrow R^n$$

by taking

$$H(x, t) = (1 - t)f(x) + tg(x)$$

for each $x \in X$ and each $t \in I$. Then we have $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for every $x \in X$.

It remains to show the continuity of H . For this purpose, it suffices to prove the continuity of the composed functions

$$p_i \circ H : X \times I \rightarrow R,$$

($i = 1, \dots, n$), because of (5.4). Since the addition and the scalar multiplication in R^n are defined coordinate-wise, we have

$$(p_i \circ H)(x, t) = (1 - t)p_i[f(x)] + tp_i[g(x)]$$

for each $x \in X$ and $t \in I$. Since addition and multiplication of real numbers are continuous, this implies the continuity of $p_i \circ H$ for every $i = 1, \dots, n$. ||

PROPOSITION 7.2. *Every map $f : X \rightarrow S^n$ from a space X into the n -sphere S^n which is not surjective is homotopic to a constant map.*

Proof. Since f is not surjective, there exists a point $p \in S^n$ such that

$$f(X) \subset S^n \setminus p.$$

According to Exercise 5F, the space $S^n \setminus p$ is homeomorphic to the Euclidean space R^n . Let

$$h : R^n \rightarrow S^n \setminus p$$

be a homeomorphism and denote by q the image $h(O)$ of the origin O in R^n .

By (7.1), the map $\phi : X \rightarrow R^n$ defined by

$$\phi(x) = h^{-1}[f(x)]$$

for every $x \in X$ is homotopic to the constant map $\psi : X \rightarrow R^n$ defined by $\psi(x) = 0$. Hence, there exists a homotopy

$$\{\theta_t : X \rightarrow R^n \mid t \in I\}$$

such that $\theta_0 = \phi$ and $\theta_1 = \psi$.

Define a homotopy $\{f_t : X \rightarrow S^n \mid t \in I\}$ by taking

$$f_t(x) = h[\theta_t(x)]$$

for each $x \in X$ and each $t \in I$. Then $f_0 = f$ and $f_1(X) = q$. ||

PROPOSITION 7.3. *The relation $\simeq$ between the maps of X into Y is an equivalence relation.*

Proof. It suffices to verify that $\simeq$ is reflexive, symmetric, and transitive. Let $f : X \rightarrow Y$ be any map and define a family $\{h_t : X \rightarrow Y \mid t \in I\}$ of maps by taking $h_t = f$ for every $t \in I$. Then one can easily verify that $\{h_t\}$ is a homotopy with $h_0 = f$ and $h_1 = f$. Hence $f \simeq f$.

Next, let $f, g : X \rightarrow Y$ be two maps with $f \simeq g$. Then there exists a homotopy $\{h_t\}$ with $h_0 = f$ and $h_1 = g$. Define a family $\{k_t\}$ of maps by taking $k_t = h_{1-t}$ for each $t \in I$. Then one can easily verify that $\{k_t\}$ is a homotopy with $k_0 = g$ and $k_1 = f$. Hence $g \simeq f$.

Finally, let $f \simeq g$ and $g \simeq h$. Then there exist homotopies $\xi_t, \eta_t : X \rightarrow Y$, ($t \in I$), such that $\xi_0 = f$, $\xi_1 = g = \eta_0$, and $\eta_1 = h$. Define a family $\{\zeta_t : X \rightarrow Y \mid t \in I\}$ of maps by taking

$$\zeta_t = \begin{cases} \xi_{2t}, & (0 \leq t \leq \frac{1}{2}), \\ \eta_{2t-1}, & (\frac{1}{2} \leq t \leq 1). \end{cases}$$

Using (4.3), one can verify that $\{\zeta_t\}$ is a homotopy. Since $\zeta_0 = \xi_0 = f$ and $\zeta_1 = \eta_1 = h$, we have $f \simeq h$. ||

It follows from (7.3) that the maps from X into Y are divided into disjoint equivalence classes, called the *homotopy classes*.

If $f, g : X \rightarrow Y$ are imbeddings, then we may define a stronger relation than $\simeq$ as follows: Two imbeddings $f, g : X \rightarrow Y$ are said to be *isotopic* iff there exists an isotopy $\{h_t : X \rightarrow Y \mid t \in I\}$ such that $h_0 = f$ and $h_1 = g$, in symbols, $f \cong g$. The proof of the following proposition is similar to that of (7.3).

PROPOSITION 7.4. *The relation $\cong$ between the imbeddings of X into Y is an equivalence relation.*

By (7.4), the imbeddings of X into Y are divided into disjoint equivalence classes, called the *isotopy classes*.

Now, let X be a subspace of a space Y . A homotopy

$$\{h_t : X \rightarrow Y \mid t \in I\}$$

is called a *deformation* iff h_0 is the inclusion map. Furthermore, if h_1 is a constant map, then the deformation $\{h_t\}$ is called a *contraction*. If a contraction $\{h_t : X \rightarrow Y \mid t \in I\}$ exists, we say that the subspace X is *contractible in the space Y* . In particular, if $X = Y$, then X is said to be *contractible*.

For example, it follows from (7.2) that every proper subspace of the n -sphere S^n is contractible in S^n . By (7.1), the Euclidean space R^n is contractible. In fact, this can be generalized to the following

PROPOSITION 7.5. *If every space X_μ in an indexed family*

$$\mathcal{F} = \{X_\mu \mid \mu \in M\}$$

of nonempty spaces is contractible, then so is the topological product

$$\Phi = \prod_{\mu \in M} X_\mu.$$

Proof. Let $\mu \in M$. Since the space X_μ is contractible, there exists a map

$$h_\mu : X_\mu \times I \rightarrow X_\mu$$

such that $h_\mu(x, 0) = x$ for every $x \in X_\mu$ and that $h_\mu(X_\mu \times 1)$ is a single point $a_\mu \in X_\mu$. Consider the Cartesian product

$$H = \prod_{\mu \in M} h_\mu : \Phi \times I^M \rightarrow \Phi$$

of the indexed family

$$\mathcal{H} = \{h_\mu \mid \mu \in M\}$$

of maps. By (5.7), H is continuous.

Next, form the Cartesian product

$$K = i \times d : \Phi \times I \rightarrow \Phi \times I^M$$

of the identity map $i : \Phi \rightarrow \Phi$ and the diagonal injection $d : I \rightarrow I^M$. By (5.5) and (5.7), K is continuous.

Finally, let us investigate the composed map

$$h = H \circ K : \Phi \times I \rightarrow \Phi.$$

One can easily verify that $h(f, 0) = f$ for every $f \in \Phi$ and that $h(\Phi \times 1)$ is the point $a \in \Phi$ defined by

$$a(\mu) = a_\mu, \quad (\mu \in M).$$

Hence h defines a contraction of Φ . ||

Next, let A be a subspace of a space X . If a deformation

$$\{h_t : X \rightarrow X \mid t \in I\}$$

exists such that h_1 is a retraction of X onto A , then the subspace A is called a *deformation retract* of X . If $h_t \mid A$ is the inclusion map for all $t \in I$, then A is said to be a *strong deformation retract* of X . As an example of deformation retracts, we have the following

PROPOSITION 7.6. *If X is a contractible space and if A is a subspace of X which consists of a single point $x_0 \in X$, then A is a deformation retract of X .*

Proof. Since X is contractible, there exists a deformation

$$\{d_t : X \rightarrow X \mid t \in I\}$$

such that $d_1(X)$ is a point $x_1 \in X$. Define a homotopy

$$\{h_t : X \rightarrow X \mid t \in I\}$$

by taking

$$h_t(x) = \begin{cases} d_{2t}(x), & (\text{if } 0 \leq t \leq \frac{1}{2}), \\ d_{2-2t}(x_0), & (\text{if } \frac{1}{2} \leq t \leq 1) \end{cases}$$

for every $x \in X$. Then $\{h_t\}$ is a deformation with $h_1(X) = x_0$. Hence the subspace $A = \{x_0\}$ is a deformation retract of X . $\square$

A map $f : X \rightarrow Y$ is said to be a *homotopy equivalence* iff there exists a map $g : Y \rightarrow X$ such that the composed maps

$$g \circ f : X \rightarrow X, \quad f \circ g : Y \rightarrow Y$$

are homotopic to the identity maps on X and Y respectively. In this case, the map $g : Y \rightarrow X$ is also a homotopy equivalence.

Obviously, every homeomorphism is a homotopy equivalence. As another example of homotopy equivalences, we have the following

PROPOSITION 7.7. *If a subspace A of a space X is a deformation retract of X , then the inclusion map $i : A \subset X$ is a homotopy equivalence.*

Proof. Since A is a deformation retract of X , there exists a deformation

$$\{d_t : X \rightarrow X \mid t \in I\}$$

such that d_1 is a retraction of X onto A . Hence, d_1 defines a map $r : X \rightarrow A$. Then, $r \circ i$ is the identity map on A . On the other hand, we have $d_1 = i \circ r$ and hence $i \circ r$ is homotopic to the identity map on X . This proves that $i : A \subset X$ is a homotopy equivalence. $\square$

An imbedding $f: X \rightarrow Y$ is said to be an *isotopy equivalence* iff there exists an imbedding $g: Y \rightarrow X$ such that the composed imbeddings

$$g \circ f: X \rightarrow X, \quad f \circ g: Y \rightarrow Y$$

are isotopic to the identity maps on X and Y respectively. In this case, the imbedding $g: Y \rightarrow X$ is also an isotopy equivalence.

Clearly, every isotopy equivalence is a homotopy equivalence, but the converse is false.

Obviously, every homeomorphism is an isotopy equivalence. As another example of isotopy equivalences, we have the following

PROPOSITION 7.8. *If J denotes the open unit interval defined by*

$$J = \{t \in I \mid 0 < t < 1\},$$

then the inclusion map $i: J \subset I$ is an isotopy equivalence.

Proof. Define an imbedding $j: I \rightarrow J$ by taking

$$j(x) = \frac{1}{3}(x + 1)$$

for every $x \in I$. It suffices to prove that the composed imbeddings

$$i \circ j: I \rightarrow I, \quad j \circ i: J \rightarrow J$$

are isotopic to the identity maps on I and J respectively.

For this purpose, let us define an isotopy

$$\{f_t: I \rightarrow I \mid t \in I\}$$

by taking

$$f_t(x) = \frac{1}{3}(3x + t - 2tx)$$

for each $t \in I$ and each $x \in I$. Clearly, f_0 is the identity map on I and $f_1 = i \circ j$. This proves that $i \circ j$ is isotopic to the identity map on I .

On the other hand, we observe that

$$f_t(J) \subset J, \quad (t \in I).$$

Hence, $\{f_t \mid t \in I\}$ defines an isotopy

$$\{g_t: J \rightarrow J \mid t \in I\}.$$

Since clearly g_0 is the identity map on J and $g_1 = j \circ i$, this proves that $j \circ i$ is isotopic to the identity map on J . ||

If a homotopy equivalence $h: X \rightarrow Y$ exists, we say that X is *homotopically equivalent* to Y , in symbols,

$$X \simeq Y.$$

Similarly, if an isotopy equivalence $i : X \rightarrow Y$ exists, X is said to be *isotopically equivalent* to Y , in symbols,

$$X \cong Y.$$

One can easily verify that the relations $\simeq$ and $\cong$ between spaces are both equivalence relations.

By (7.6) and (7.7), any two contractible spaces are homotopically equivalent; in particular, any two Euclidean spaces are homotopically equivalent. By (7.8), the closed interval I and the open interval J are isotopically equivalent.

A property of spaces is called a *homotopy property* iff it is preserved by every homotopy equivalence. In other words, a homotopy property is one which when possessed by a space X is also possessed by every space Y homotopically equivalent to X . Similarly, a property which is preserved by every isotopy equivalence will be called an *isotopy property*.

Since every homeomorphism is an isotopy equivalence and every isotopy equivalence is a homotopy equivalence, it follows that every homotopy property is an isotopy property and every isotopy property is a topological property.

As an example of homotopy properties, we have the following

PROPOSITION 7.9. *The contractibility of a space is a homotopy property.*

Proof. Assume that X is contractible and that $X \simeq Y$. We will prove that Y is also contractible.

Since X is contractible, there exists a contraction

$$\{c_t : X \rightarrow X \mid t \in I\}.$$

Next, since $X \simeq Y$, there exist maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g$ is homotopic to the identity map on Y . Let

$$\{d_t : Y \rightarrow Y \mid t \in I\}$$

be a deformation with $d_1 = f \circ g$. Define a homotopy

$$\{h_t : Y \rightarrow Y \mid t \in I\}$$

by taking

$$h_t = \begin{cases} d_{2t}, & (\text{if } 0 \leq t \leq \frac{1}{2}), \\ f \circ c_{2t-1} \circ g, & (\text{if } \frac{1}{2} \leq t \leq 1). \end{cases}$$

Since $h_0 = d_0$ is the identity map on Y and $h_1 = f \circ c_1 \circ g$ is a constant map, $\{h_t\}$ is a contraction. Hence Y is contractible. ||

In the next chapter, we shall study more homotopy properties, isotopy properties, and topological properties of spaces.

EXERCISES

- 7A. Prove that the unit n -sphere S^n in the Euclidean space R^{n+1} is a strong deformation retract of $R^{n+1} \setminus O$, where O denotes the origin of R^{n+1} .
- 7B. Prove that the cone $\text{Con}(X)$ over an arbitrary space X is contractible.
- 7C. Prove that a map $f: X \rightarrow Y$ is homotopic to a constant map iff f has a continuous extension over $\text{Con}(X)$.
- 7D. Prove that, for an arbitrary map $f: X \rightarrow Y$, the subspace Y of the mapping cylinder $\text{Cyl}(f)$ is a strong deformation retract of $\text{Cyl}(f)$.
- 7E. Prove that, for an arbitrary set M , the topological powers I^M and R^M are isotopically equivalent.
- 7F. Prove that every two indiscrete spaces are homotopically equivalent. Hence the cardinal number of the points in a space is not a homotopy property.

Chapter III: PROPERTIES OF SPACES AND MAPS

As the title indicates, this chapter is devoted to the properties of spaces and maps usually studied in topology. Since there are too many of these properties to be all included in a single chapter, we had to select the properties that would be important enough and could be treated in a desirably leisurely fashion. Among those thus excluded from this chapter, some are topological properties, such as metrizability and uniformizability; others are not topological properties, for example, completeness and completions of metric spaces or uniform spaces. These will be studied in the next two chapters.

1. SEPARATION AXIOMS

By a *Hausdorff space*, we mean a space X which satisfies the following *Hausdorff separation axiom*: Every pair of distinct points of X has disjoint neighborhoods in X ; in other words, if a and b are distinct points of X , then there exist open sets U and V in X such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$.

For example, the real line R is a Hausdorff space. Every discrete space clearly satisfies the Hausdorff separation axiom, while an indiscrete space with more than one point can never be a Hausdorff space.

PROPOSITION 1.1. *Every subspace E of a Hausdorff space X is a Hausdorff space.*

Proof. Let a, b be any two distinct points in E . Since X is a Hausdorff space, there exist open sets U and V in X such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$. Let $U^* = E \cap U$ and $V^* = E \cap V$. Then U^* and V^* are open sets of the subspace E . Since $a \in U^*$, $b \in V^*$ and $U^* \cap V^* = \emptyset$, it follows that E is a Hausdorff space. $\square$

PROPOSITION 1.2. *The topological sum X of any disjoint collection $\{X_\mu \mid \mu \in M\}$ of Hausdorff spaces is a Hausdorff space.*

Proof. Let a, b be any two distinct points in X . Then there exist $\mu, \nu \in M$ such that $a \in X_\mu$ and $b \in X_\nu$. If $\mu \neq \nu$, then take $U = X_\mu$ and $V = X_\nu$. If $\mu = \nu$, then a and b are two distinct points in a Hausdorff space X_μ and hence there are open sets U and V in X_μ such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$. In both cases, U and V are open sets of the topological sum X . This implies that X is a Hausdorff space. $\square$

PROPOSITION 1.3. *The topological product X of any collection $\{X_\mu \mid \mu \in M\}$ of Hausdorff spaces is a Hausdorff space.*

Proof. Let a, b be any two distinct points in X . By definition of X , $a \neq b$ implies the existence of a $\nu \in M$ such that $a(\nu)$ and $b(\nu)$ are distinct points in X_ν . Since X_ν is a Hausdorff space, there exist open sets U and V in X_ν such that $a(\nu) \in U$, $b(\nu) \in V$, and $U \cap V = \emptyset$. Let U^* and V^* denote the sub-basic open sets in X defined by

$$\begin{aligned} U^* &= \{x \in X \mid x(\nu) \in U\}, \\ V^* &= \{x \in X \mid x(\nu) \in V\}. \end{aligned}$$

Then, $a \in U^*$, $b \in V^*$, and $U^* \cap V^* = \emptyset$. This implies that X is a Hausdorff space. $\square$

As a consequence of (1.3), the n -dimensional Euclidean space R^n and the Hilbert space R^∞ are Hausdorff spaces. Hence, by (1.1), the unit n -simplex Δ^n and the unit n -sphere S^n are Hausdorff spaces.

PROPOSITION 1.4. *If E is a retract of a Hausdorff space X , then E is a closed set in X .*

Proof. Assume that E is a retract of a Hausdorff space X with a retraction $r: X \rightarrow E$. We are going to prove that $X \setminus E$ is an open set in X .

If $X \setminus E = \emptyset$, the proposition is trivial. Thus, we assume that $X \setminus E \neq \emptyset$. Let a be any point in $X \setminus E$. Then the point $b = r(a)$ is in E and hence $a \neq b$. Since X is a Hausdorff space, there exist open sets U and V of X such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$. Since r is continuous and $E \cap V$ is an open set in E , it follows that $r^{-1}(E \cap V)$ is an open set of X containing a . Let

$$W = U \cap r^{-1}(E \cap V).$$

Then W is an open set of X such that $a \in W$, $V \cap W = \emptyset$, and $r(W) \subset V$. This implies that $r(x) \neq x$ for every $x \in W$ and hence $W \subset X \setminus E$. This proves that $X \setminus E$ is open and, therefore, E is closed. $\square$

If we apply (1.4) to the special case where E consists of a single point, we deduce that a Hausdorff space X has the following property: *Every point of X forms a closed set in X .* A space X which has this property is called a *Fréchet space* or a T_1 -space. It follows immediately that every Fréchet space X satisfies the following *Fréchet separation axiom*: For any pair of distinct points a and b in X , there exists an open set U of X such that $a \in U$ and $b \notin U$.

A space X is said to be *regular at a point p of X* iff every neighborhood of p contains a closed neighborhood of p . By a *regular space*, we mean a space X which is regular at each of its points.

PROPOSITION 1.5. *Every regular Fréchet space is a Hausdorff space.*

Proof. Let X be a regular Fréchet space and a, b be any two distinct points in X . Since X is a Fréchet space, $X \setminus b$ is an open set of X . From the regularity of X at the point a , there follows the existence of a closed neighborhood N of a with $N \subset X \setminus b$. Let

$$U = \text{Int}(N), \quad V = X \setminus N.$$

Then U and V are open sets of X such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$. $\square$ $\parallel$

A space X is said to be *completely regular at a point p of X* iff, for every neighborhood N of p in X , there exists a continuous function

$$\chi: X \rightarrow I$$

such that $\chi(p) = 0$ and $\chi(X \setminus N) = 1$. Since the set

$$M = \{x \in X \mid \chi(x) \leq \tfrac{1}{2}\}$$

is clearly a closed neighborhood of p contained in N , it follows that X is also regular at the point p . By a *completely regular space*, we mean a space X which is completely regular at each of its points. Hence, every completely regular space is regular. A completely regular Fréchet space is called a *Tychonoff space*.

A space X is said to be *normal* iff every disjoint pair of closed sets in X has disjoint neighborhoods; in other words, for any two disjoint closed sets A and B in X , there exist open sets U and V in X such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Discrete spaces and indiscrete spaces are normal. Hence, a normal space may fail to satisfy the Fréchet axiom of separation or the first or second axiom of countability. On the other hand, since singletons are

closed in a Fréchet space, it follows that every normal Fréchet space is regular and hence also a Hausdorff space.

THEOREM 1.6 (URYSOHN'S LEMMA). *If A and B are two disjoint closed sets in a normal space X , then there exists a continuous function*

$$\chi: X \rightarrow I$$

such that $\chi(A) = 0$ and $\chi(B) = 1$.

Proof. Let D denote the set of all non-negative dyadic rational numbers, that is to say, the set of all numbers of the form $a/2^q$, where a and q are non-negative integers.

First of all, we will construct an indexed family $F = \{U_t \mid t \in D\}$ of open sets in X such that, for any two distinct $s, t \in D$, $s < t$ implies that

$$\text{Cl}(U_s) \subset U_t.$$

For this purpose, we define $U_t = X$ for every $t > 1$ in D and set $U_1 = X \setminus B$. Since X is normal and A, B are disjoint, there exist two open sets M and N of X such that $A \subset M$, $B \subset N$, and $M \cap N = \emptyset$. Then we define $U_0 = M$. This implies that

$$\text{Cl}(U_0) \subset X \setminus N \subset X \setminus B = U_1.$$

Next, let $t \in D$ be satisfying $0 < t < 1$. Then t can be uniquely written in the form

$$t = (2m + 1)/2^n.$$

We will construct U_t by induction on n . Let

$$\begin{aligned}\alpha &= 2m/2^n = m/2^{n-1}, \\ \beta &= (2m + 2)/2^n = (m + 1)/2^{n-1}.\end{aligned}$$

Then $\alpha < t < \beta$ and, in accordance to the inductive hypothesis, the open sets U_α and U_β have already been constructed in such a way that $\text{Cl}(U_\alpha) \subset U_\beta$. Hence, $\text{Cl}(U_\alpha)$ and $X \setminus U_\beta$ are two disjoint closed sets in X . Since X is normal, there exist two open sets V and W of X such that

$$\text{Cl}(U_\alpha) \subset V, \quad X \setminus U_\beta \subset W, \quad V \cap W = \emptyset.$$

Let $U_t = V$. Then we obtain

$$\text{Cl}(U_\alpha) \subset U_t, \quad \text{Cl}(U_t) \subset X \setminus W \subset U_\beta.$$

This completes the inductive construction of the indexed family $F = \{U_t \mid t \in D\}$ of open sets in X . From the construction, it is clear that F covers X .

Now, define a real function

$$\chi : X \rightarrow I$$

by taking $\chi(x) = \inf \{t \mid x \in U_t\}$ for each $x \in X$. Then we have

$$\chi(A) = 0, \quad \chi(B) = 1.$$

Hence it remains to verify the continuity of this function χ .

For each real number $a \in I$, let

$$L_a = \{t \in I \mid t < a\}, \quad R_a = \{t \in I \mid t > a\}.$$

Then the collection $\{L_a, R_a \mid a \in I\}$ forms a sub-basis of the topology of I . It suffices to prove that, for each $a \in I$, $\chi^{-1}(L_a)$ and $\chi^{-1}(R_a)$ are open in X .

First, let us consider the set

$$\chi^{-1}(L_a) = \{x \in X \mid \chi(x) < a\}.$$

Since $\chi(x) = \inf \{t \mid x \in U_t\}$ and since the infimum is less than a iff some member of $\{t \mid x \in U_t\}$ is less than a , the set $\chi^{-1}(L_a)$ consists of all points $x \in X$ such that $x \in U_t$ for some $t < a$. Hence

$$\chi^{-1}(L_a) = \bigcup \{U_t \mid t \in D \text{ and } t < a\}.$$

This implies that $\chi^{-1}(L_a)$ is an open set of X .

Finally, in order to show that $\chi^{-1}(R_a)$ is open, it suffices to prove that the set

$$\chi^{-1}(I \setminus R_a) = \{x \in X \mid \chi(x) \leq a\}$$

is closed. Since $\chi(x) = \inf \{t \mid x \in U_t\}$, it follows that $\chi(x) \leq a$ iff $x \in U_t$ for every $t > a$ in D . Hence

$$\chi^{-1}(I \setminus R_a) = \bigcap \{U_t \mid t \in D \text{ and } t > a\}.$$

To complete the proof, it remains to prove that

$$\chi^{-1}(I \setminus R_a) = \bigcap \{\text{Cl}(U_t) \mid t \in D \text{ and } t > a\}.$$

For this purpose, let r be any member of D with $r > a$. Since D is dense in I , there is an $s \in D$ such that $a < s < r$ and hence $\text{Cl}(U_s) \subset U_r$. Since $s > a$, we deduce

$$\bigcap \{\text{Cl}(U_t) \mid t \in D \text{ and } t > a\} \subset \text{Cl}(U_s) \subset U_r.$$

Since r is an arbitrary member of D with $r > a$, this implies that

$$\bigcap \{\text{Cl}(U_t) \mid t \in D \text{ and } t > a\} \subset \chi^{-1}(I \setminus R_a).$$

Since the inclusion

$$\chi^{-1}(I \setminus R_a) \subset \bigcap \{ \text{Cl}(U_t) \mid t \in D \text{ and } t > a \}$$

is obvious, this completes the proof of (1.6). $\parallel$

COROLLARY 1.7. *Every normal Fréchet space is a Tychonoff space.*

Proof. Let X be a normal Fréchet space and p be an arbitrary point in X . It remains to prove that X is completely regular at p . For this purpose, let N be any neighborhood of p in X and let $U = \text{Int}(N)$. Then

$$A = \{p\}, \quad B = X \setminus U$$

are two disjoint closed sets in X . By (1.6), there exists a continuous function $\chi: X \rightarrow I$ such that

$$\chi(p) = \chi(A) = 0,$$

$$\chi(X \setminus N) \subset \chi(X \setminus U) = \chi(B) = 1.$$

Hence X is completely regular at p . $\parallel$

EXERCISES

- 1A. If X is a Hausdorff space and $x_1, \dots, x_n$ are n distinct points in X , prove that there exist n disjoint open sets $U_1, \dots, U_n$ in X such that $x_i \in U_i$ for every $i = 1, \dots, n$. Hence, every finite Hausdorff space is discrete.
- 1B. Prove the following assertions:
 - (a) Every subspace of a Fréchet space is a Fréchet space.
 - (b) Every topological sum of a disjoint collection of Fréchet spaces is a Fréchet space.
 - (c) Every topological product of Fréchet spaces is a Fréchet space. Establish similar assertions for regular spaces and completely regular spaces.
- 1C. Prove that a quotient space Q of a space X is a Fréchet space iff every member of Q is a closed set in X .
- 1D. Prove that every closed subspace of a normal space is normal.
- 1E. Prove that, for any space X , the following statements are equivalent:
 - (a) X is a normal space.
 - (b) If an open set G of X contains a closed set F of X , then there exists an open set H of X such that $F \subset H$ and $\text{Cl}(H) \subset G$.
 - (c) For any two disjoint closed sets A and B in X , there are open

sets U and V of X such that $A \subset U$, $B \subset V$, and $\text{Cl}(U) \cap \text{Cl}(V) = \square$.

- (d) For any finite number of closed sets $F_1, \dots, F_n$ of X such that $F_1 \cap \dots \cap F_n = \square$, there exist open sets $U_1, \dots, U_n$ of X such that $F_i \subset U_i$ for all $i = 1, \dots, n$ and $U_1 \cap \dots \cap U_n = \square$.
- (e) For any finite number of open sets $G_1, \dots, G_n$ of X such that $G_1 \cup \dots \cup G_n = X$, there exist open sets $H_1, \dots, H_n$ of X such that $\text{Cl}(H_i) \subset G_i$ for all $i = 1, \dots, n$ and $H_1 \cup \dots \cup H_n = X$.

- 1F. Let X denote the union of the set R of all real numbers and a singleton set $\{\infty\}$. Give X the smallest topology which contains every open set U of the real line R and every set $V \subset X$ such that $X \setminus V$ is a finite set contained in R . Prove that the space X thus obtained is a Fréchet space but not a Hausdorff space.
- 1G. Let X denote the set $\{t \in R \mid 0 \leq t \leq 1\}$ and D the subset $\{1/n \mid n = 1, 2, \dots\}$. Give X the smallest topology which contains every open set of $X \setminus \{0\}$ as a subspace of the real line R and every set B_a , $(0 < a \leq 1)$, defined by

$$B_a = \{t \in X \mid t < a \text{ and } t \notin D\}.$$

Prove that the space X thus obtained is a Hausdorff space but not a regular space.

- 1H. Prove that every neighborhood of an accumulation point of a set E in a Fréchet space X contains infinitely many points of E .
- 1I. Prove that the topological product of a family of Tychonoff spaces is a Tychonoff space.
- 1J. A space X is said to be *completely normal* iff every subspace of X is normal. Prove that a space X is completely normal iff, for any two sets A and B such that $A \cap \text{Cl}(B) = \square = \text{Cl}(A) \cap B$, then there exist two disjoint open sets U and V in X with $A \subset U$ and $B \subset V$.

2. COMPACTNESS

A space X is said to be *compact* iff every open cover of X has a finite subcover. A set K in a space X is said to be *compact* iff, with the relative topology, the subspace K is compact; equivalently, a set K in X is compact iff every cover of K by open sets in X has a finite subcover.

The definition of compactness given above was motivated by the Heine-Borel theorem for bounded closed sets of the real line R as given

in any standard book on real analysis. Hence, every bounded closed set of the real line R is compact; in particular, the unit interval I is compact.

A family $\mathcal{F}$ of sets is said to have the *finite intersection property* iff the intersection of the members of each finite subfamily of $\mathcal{F}$ is nonempty.

PROPOSITION 2.1. *A space X is compact iff every family of closed sets in X which has the finite intersection property has a nonempty intersection.*

Proof: Necessity. Let $\mathcal{F}$ be any family of closed sets in a compact space X and assume that the intersection of all members of $\mathcal{F}$ is empty. Since the complement of the intersection is equal to the union of the complements according to De Morgan formulae, it follows that the collection $\mathcal{G} = \{X \setminus A \mid A \in \mathcal{F}\}$ is an open cover of the compact space X . By the definition of compactness, $\mathcal{G}$ has a finite subcover. Hence, there are a finite number of sets $A_1, \dots, A_n$ in $\mathcal{F}$ such that $\{X \setminus A_1, \dots, X \setminus A_n\}$ covers X . By De Morgan's formulae, this implies $A_1 \cap \dots \cap A_n = \square$. Hence $\mathcal{F}$ does not have the finite intersection property.

Sufficiency. Let $\mathcal{C}$ be any open cover of X and consider the family $\mathcal{F} = \{X \setminus U \mid U \in \mathcal{C}\}$ of closed sets. Since $\mathcal{C}$ is a cover of X , it follows from De Morgan formulae that the intersection of all members of $\mathcal{F}$ is empty. Hence, $\mathcal{F}$ does not have the finite intersection property; in other words, there are a finite number of open sets $U_1, \dots, U_n$ in $\mathcal{C}$ such that $(X \setminus U_1) \cap \dots \cap (X \setminus U_n) = \square$. By De Morgan formulae, this implies that $\{U_1, \dots, U_n\}$ is a finite subcover of X in $\mathcal{C}$. Hence X is compact. ||

PROPOSITION 2.2. *Every closed set K in a compact space X is compact.*

Proof. Let $\mathcal{C}$ be a cover of K by open sets of X . Since K is closed, the complement $X \setminus K$ is open. Hence, the members of $\mathcal{C}$ together with the open set $X \setminus K$ form an open cover of X . Since X is compact, this open cover of X contains a finite subcover of X . In other words, there are a finite number of open sets $U_1, \dots, U_n$ in $\mathcal{C}$ such that

$$U_1 \cup \dots \cup U_n \cup X \setminus K = X.$$

Hence $\{U_1, \dots, U_n\}$ covers K and K is compact. ||

PROPOSITION 2.3. *Every continuous image of a compact space is compact.*

Proof. Let $f: X \rightarrow Y$ be a surjective map from a compact space X onto a space Y . We will prove that Y is compact. For this purpose, let $\mathcal{C}$ be any open cover of Y and consider

$$\mathcal{B} = \{f^{-1}(V) \mid V \in \mathcal{C}\}.$$

Since f is continuous, $\mathcal{B}$ is an open cover of X . Since X is compact, $\mathcal{B}$ has a finite subcover; in other words, there are a finite number of open sets $V_1, \dots, V_n$ in $\mathcal{C}$ such that

$$X = f^{-1}(V_1) \cup \dots \cup f^{-1}(V_n).$$

Since f is surjective, we have $f[f^{-1}(V_i)] = V_i$ for every $i = 1, \dots, n$. Hence

$$Y = f(X) = V_1 \cup \dots \cup V_n.$$

This implies that Y is compact. $\square$

PROPOSITION 2.4. *If K is a compact set in a Hausdorff space X and p is a point in $X \setminus K$, then there are disjoint open sets U and V of X such that $K \subset U$ and $p \in V$.*

Proof. For each point $a \in K$, there exist disjoint open sets U_a and V_a of X such that $a \in U_a$ and $p \in V_a$ since X is a Hausdorff space and $a \neq p$. Then the family $\mathcal{F} = \{U_a \mid a \in K\}$ is a cover of K by open sets of X . Since K is compact, $\mathcal{F}$ has a finite subcover of K . Hence, there are a finite number of points $a_1, \dots, a_n$ of K such that K is contained in the union

$$U = U_{a_1} \cup \dots \cup U_{a_n}.$$

On the other hand, let

$$V = V_{a_1} \cap \dots \cap V_{a_n}.$$

Then U and V are open sets of X such that $K \subset U$, $p \in V$, and $U \cap V = \emptyset$. $\square$

COROLLARY 2.5. *Every compact set K in a Hausdorff space X is closed.*

Proof. For each $p \in X \setminus K$, the open set V in (2.4) is contained in

$$X \setminus U \subset X \setminus K.$$

Hence, $X \setminus K$ is open and K is closed. $\square$

PROPOSITION 2.6. *Every bijective map $f: X \rightarrow Y$ from a compact space X onto a Hausdorff space Y is a homeomorphism.*

Proof. Since f is bijective and continuous, it remains to prove that f is closed. For this purpose, let E be any closed set in X . By (2.2), E is compact. By (2.3), $f(E)$ is compact. By (2.5), $f(E)$ is closed. Hence f is closed. $\square$

PROPOSITION 2.7. *Every compact Hausdorff space is normal.*

Proof. Let A and B be any two disjoint closed sets in a compact

Hausdorff space X . By (2.2), A and B are compact. According to (2.4), we may pick for each $b \in B$ two disjoint open sets U_b and V_b of X such that $A \subset U_b$ and $b \in V_b$. Then the collection

$$\mathcal{F} = \{V_b \mid b \in B\}$$

is a cover of B by open sets of X . Since B is compact, $\mathcal{F}$ has a finite subcover of B . In other words, there are a finite number of points $b_1, \dots, b_n$ in B such that B is contained in the union

$$V = V_{b_1} \cup \dots \cup V_{b_n}.$$

On the other hand, let

$$U = U_{b_1} \cap \dots \cap U_{b_n}.$$

Then U and V are open sets of X such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. Hence X is normal. $\square$

PROPOSITION 2.8. *If K is a compact set in a regular space X and if U is a neighborhood of K in X , then U contains a closed neighborhood V of K .*

Proof. For each point p in K , there exists an open neighborhood V_p of p in X such that $\text{Cl}(V_p) \subset U$ since X is regular. Then the family

$$\mathcal{F} = \{V_p \mid p \in K\}$$

is a cover of K by open sets of X . Since K is compact, there exist a finite number of points $p_1, \dots, p_n$ of K such that K is contained in the union of V_{p_i} , $i = 1, 2, \dots, n$. Let

$$V = \text{Cl}(V_{p_1}) \cup \dots \cup \text{Cl}(V_{p_n}).$$

Then V is a closed neighborhood of K contained in U . $\square$

COROLLARY 2.9. *Every compact regular space is normal.*

Proof. Let A and B be any two disjoint closed sets in a compact regular space X . By (2.2), A is compact. Since $X \setminus B$ is an open neighborhood of A in X , it follows from (2.8) that A has a closed neighborhood N contained in $X \setminus B$. Let

$$U = \text{Int}(N), \quad V = X \setminus N.$$

Then U and V are open sets of X such that

$$A \subset U, \quad B \subset V, \quad U \cap V = \emptyset.$$

Hence X is normal. $\square$

THEOREM 2.10 (TYCHONOFF'S THEOREM). *The topological product of a family of compact spaces is compact.*

In the proof of this theorem, we will make use of a set-theoretic notion as follows. A class $\mathcal{C}$ of sets is said to be of finite character iff the following condition is satisfied: a set E is a member of $\mathcal{C}$ iff every finite subset of E belongs to $\mathcal{C}$. An equivalent version of the Axiom of Choice is the Tukey Lemma: *For each member A of a class $\mathcal{C}$ of sets which is of finite character, there exists a maximal member M in $\mathcal{C}$ containing A .* [K, p. 33.]

Proof of (2.10). Let $\mathcal{F} = \{X_\mu \mid \mu \in M\}$ be a family of compact spaces and let X denote the topological product of the family $\mathcal{F}$. We are going to prove the compactness of X . Because of (2.1), it suffices to establish the assertion stated as follows: If $\mathcal{B}$ is a family of subsets of X having the finite intersection property, then

$$\bigcap \{\text{Cl}(B) \mid B \in \mathcal{B}\} \neq \emptyset.$$

For this purpose, let us consider the class $\mathcal{C}$ of all families of subsets of X which possess the finite intersection property. This class $\mathcal{C}$ clearly is of finite character. Hence it follows from the Tukey Lemma that there is a maximal member of $\mathcal{C}$ which contains the given family $\mathcal{B}$. Without loss of generality, we may assume that $\mathcal{B}$ itself is a maximal member of the class $\mathcal{C}$.

From the maximality of $\mathcal{B}$, it follows that the intersection of the members of every finite subfamily of $\mathcal{B}$ belongs to $\mathcal{B}$. Besides, if a subset E of X meets every member of $\mathcal{B}$, then E is also a member of $\mathcal{B}$.

Now, let $\mu \in M$ and consider the natural projection

$$p_\mu: X \rightarrow X_\mu.$$

Consider the family

$$\mathcal{B}_\mu = \{p_\mu(B) \mid B \in \mathcal{B}\}$$

of subsets of X_μ . Then $\mathcal{B}_\mu$ has the finite intersection property. Since X_μ is compact, it follows from (2.1) that the intersection

$$H_\mu = \bigcap \{\text{Cl}[p_\mu(B)] \mid B \in \mathcal{B}\}$$

is nonempty. Pick a point $x_\mu \in H_\mu$ for each index $\mu \in M$.

Let x denote the point of the topological product X whose μ -th coordinate $x(\mu)$ is the point $x_\mu \in X_\mu$. It remains to prove that $x \in \text{Cl}(B)$ for every $B \in \mathcal{B}$.

For this purpose, consider an arbitrary sub-basic open set U_μ^* of X which contains the point x , where $\mu \in M$ and U_μ is an open set of X_μ containing the point x_μ . Since $x_\mu \in \text{Cl}[p_\mu(B)]$ for each $B \in \mathcal{B}$, it follows that U_μ meets $p_\mu(B)$ for each $B \in \mathcal{B}$. Hence, the set

$$U_\mu^* = p_\mu^{-1}(U_\mu)$$

meets every member B of $\mathcal{B}$. It follows from the maximality of $\mathcal{B}$ that U_μ^* belongs to $\mathcal{B}$ and, therefore, the intersection of any finite number of sub-basic open sets containing x is a member of $\mathcal{B}$. This implies that every neighborhood of x in X meets each member B of $\mathcal{B}$. Consequently, $x \in \text{Cl}(B)$ for every $B \in \mathcal{B}$. ||

COROLLARY 2.11. *Every topological power I^M of the closed unit interval I is a compact Hausdorff space.*

COROLLARY 2.12. *Every bounded closed set of the n -dimensional Euclidean space R^n is compact.*

A space X is said to be *locally compact* at a point $p \in X$ iff p has at least one compact neighborhood in X . If X is locally compact at every point, it is called a *locally compact space*. Examples of locally compact spaces are compact spaces, discrete spaces, and the n -dimensional Euclidean space R^n .

PROPOSITION 2.13. *Every closed subspace of a locally compact space is locally compact.*

Proof. Let E be a closed subspace of a locally compact space X and let p be an arbitrary point in E . Since X is locally compact at p , there is a compact neighborhood K of p in X . Let $H = K \cap E$. As a closed subset of K , H is compact. Since K is a neighborhood of p in X , it follows that H is a neighborhood of p in E . This implies that E is locally compact at p . ||

PROPOSITION 2.14. *If X is a locally compact space which is either Hausdorff or regular, then the family of closed compact neighborhoods of a point $p \in X$ is a local basis at p .*

Proof. Since X is locally compact, p has a compact neighborhood K in X . Let N be an arbitrary neighborhood of p in X ; we are going to prove the existence of a closed compact neighborhood M of p contained in N .

First, assume that X is regular. There exists a closed neighborhood M in X contained in $K \cap N$. As a closed subset of a compact space K , M is also compact.

Next, assume X to be a Hausdorff space. By (2.7), K is a normal Hausdorff space and hence is also regular. Since $K \cap N$ is now a neighborhood of p in a regular space K , there is a closed neighborhood M of p in K contained in $K \cap N$. By (2.2), K is closed. Hence, M is both compact and closed in X . Since K itself is a neighborhood of p in X , it follows that M is also a neighborhood of p in X .

In both cases, M is a closed compact neighborhood of p in X contained in N . ||

COROLLARY 2.15. *Every locally compact Hausdorff space is regular.*

PROPOSITION 2.16. *If K is a closed compact set in a locally compact regular space X and if N is a neighborhood of K in X , then there exists a closed compact neighborhood M of K in X such that $K \subset M \subset N$. Furthermore, there exists a continuous function $f: X \rightarrow I$ such that*

$$f(x) = \begin{cases} 0, & (\text{if } x \in K), \\ 1, & (\text{if } x \in X \setminus M). \end{cases}$$

Proof. Let $p \in K$. By (2.14), there is a closed compact neighborhood M_p of p contained in N . Let U_p denote the interior of M_p . Then the family

$$\mathcal{F} = \{U_p \mid p \in K\}$$

is an open cover of K by open sets of X . Since K is compact, there are a finite number of points $p_1, \dots, p_n$ in K such that K is contained in the union U of the open sets $U_{p_1}, \dots, U_{p_n}$. Let

$$M = M_{p_1} \cup \dots \cup M_{p_n}.$$

Then M is closed and compact, and

$$K \subset U \subset M \subset N.$$

Hence M is a closed compact neighborhood of K contained in N . This proves the first part of the proposition.

By (2.9), the subspace M of X is normal. Since K and $M \setminus U$ are two disjoint closed sets in a normal space M , it follows from the Urysohn Lemma that there exists a continuous function $g: M \rightarrow I$ which has the value 0 on K and 1 on $M \setminus U$. Define a function $f: X \rightarrow I$ by taking

$$f(x) = \begin{cases} g(x), & (\text{if } x \in M), \\ 1, & (\text{if } x \in X \setminus M). \end{cases}$$

By (II, 4.3), f is continuous. It is obvious that $f(x) = 0$ for $x \in K$ and $f(x) = 1$ for $x \in X \setminus M$. ||

COROLLARY 2.17. *Every locally compact regular space is completely regular.*

COROLLARY 2.18. *Every locally compact Hausdorff space is a Tychonoff space.*

A family $\mathcal{F}$ of sets in a space X is said to be *locally finite* iff every

point of X has a neighborhood which meets at most a finite number of members of $\mathcal{F}$.

Let $\mathcal{F}$ and $\mathcal{G}$ be two covers of a space X . If every member of $\mathcal{F}$ is contained in some member of $\mathcal{G}$, then we say that $\mathcal{F}$ is a *refinement* of $\mathcal{G}$.

A space X is said to be *paracompact* iff every open cover of X has a locally finite open refinement, i.e. a refinement which is a locally finite open cover of X . Every compact space is paracompact. It will be seen in a later section that the converse is false. [Dieudonné 1.]

PROPOSITION 2.19. *Every paracompact Hausdorff space is regular.*

Proof. Let X be a paracompact Hausdorff space and let U be an open neighborhood of a point $p \in X$. For each point $q \in X \setminus U$, choose open sets U_q and V_q of X such that

$$q \in U_q, \quad p \in V_q, \quad U_q \cap V_q = \emptyset.$$

The open sets $\{U_q \mid q \in X \setminus U\}$ together with the given open set U form an open cover $\mathcal{C}$ of the paracompact space X . Hence, by definition, $\mathcal{C}$ has an open refinement $\mathcal{D}$ which is locally finite.

Since $\mathcal{D}$ is locally finite, there exists a neighborhood M of the point p which meets only the members of a finite subfamily $\mathcal{F}$ of $\mathcal{D}$. Let $W_1, \dots, W_n$ denote the members of $\mathcal{F}$ which are not contained in U . Then there are n points $q_1, \dots, q_n$ of $X \setminus U$ such that $W_i \subset U_{q_i}$ for each $i = 1, \dots, n$. Let

$$N = M \cap V_{q_1} \cap \dots \cap V_{q_n}, \quad V = \text{Cl}(N).$$

It remains to prove that $V \subset U$.

For this purpose, let W denote the union of all members of $\mathcal{D}$ which are not contained in U . Then we have

$$N \cap W \subset M \cap V_{q_1} \cap \dots \cap V_{q_n} \cap [U_{q_1} \cup \dots \cup U_{q_n}] = \emptyset$$

since $V_{q_i} \cap U_{q_i} = \emptyset$ for every $i = 1, \dots, n$. This implies that N is contained in the closed set $X \setminus W \subset U$. Hence

$$V = \text{Cl}(N) \subset X \setminus W \subset U. \quad ||$$

PROPOSITION 2.20. *Every paracompact regular space is normal.*

Proof. Let X be a paracompact regular space, and let A and B be any two disjoint closed sets in X . For each point $p \in A$, choose a closed neighborhood M_p of p in the space X contained in the open set $X \setminus B$. Let

$$U_p = \text{Int}(M_p).$$

The open sets $\{U_p \mid p \in A\}$ together with the open set $X \setminus A$ form an open cover $\mathcal{C}$ of the paracompact space X . Hence, by definition, $\mathcal{C}$ has an open refinement $\mathcal{D}$ which is locally finite.

Let U denote the union of all members of $\mathcal{D}$ which are not contained in $X \setminus A$. Then U is an open set of X containing A .

Let q be an arbitrary point of B . Since $\mathcal{D}$ is locally finite, there exists a neighborhood N_q of q in the space X which meets only the members of a finite subfamily $\mathcal{F}_q$ of $\mathcal{D}$. Let $D_1, \dots, D_n$ denote the members of $\mathcal{F}_q$ which are not contained in $X \setminus A$. Then, there are n points $p_1, \dots, p_n$ of A such that $D_i \subset U_{p_i}$ for each $i = 1, \dots, n$. Let

$$V_q = \text{Int}(N_q) \cap (X \setminus M_{p_1}) \cap \dots \cap (X \setminus M_{p_n}).$$

Then V_q is an open neighborhood of q such that $U \cap V_q = \emptyset$.

Let V denote the union of the open sets V_q for all $q \in B$. Then V is an open set of X containing B such that

$$U \cap V = \emptyset.$$

Hence X is a normal space. $\square$

EXERCISES

- 2A. Show that every indiscrete space is compact and normal, and that every indiscrete space of more than one point is not a Fréchet space.
- 2B. Prove that a space X is compact if its topology has a sub-basis σ such that every cover of X by members of σ has a finite subcover. Give a proof of (2.10) by using this fact.
- 2C. Let K be a compact set in a completely regular space X and let U be a neighborhood of K in X . Prove the existence of a map $f: X \rightarrow I$ such that

$$f(x) = \begin{cases} 0, & (\text{if } x \in K), \\ 1, & (\text{if } x \in X \setminus U). \end{cases}$$

- 2D. Consider spaces X, Y and compact sets $A \subset X, B \subset Y$. Let W be a neighborhood of $A \times B$ in the topological product $X \times Y$. Prove that there are neighborhoods U of A in X and V of B in Y with $U \times V \subset W$. In case $B = Y$, show that the compactness of A may be omitted from the hypothesis.
- 2E. Prove the assertion: if an infinite number of the coordinate spaces

of a topological product X are noncompact, then every compact set K in X is *nowhere dense*, that is to say, $\text{Int}(K) = \square$.

- 2F. Prove that the closure of a compact set in a regular space is compact.
- 2G. Prove the assertion: if a topology τ in a set X makes X a compact Hausdorff space, then every topology in X which is strictly smaller than τ fails to make X a Hausdorff space.
- 2H. Show that the continuous image of a locally compact space may fail to be locally compact. However, if the map is open, then the image of a locally compact space is always locally compact.
- 2I. Prove the assertion: if a topological product is locally compact, then each coordinate space is locally compact and all except a finite number of coordinate spaces are compact.
- 2J. Prove the assertion: if X is a Hausdorff space and E is a dense, locally compact subspace of X , then E is open.
- 2K. By a *Cantor space*, we mean a topological power 2^M of the discrete space 2 which consists of two elements. Prove that every compact Hausdorff space is the continuous image of a closed subspace of some Cantor space.
- 2L. Let α and β be covers of a space X . We say that β is a *star-refinement* iff, for every $B \in \beta$, there exists an $A \in \alpha$ which contains every member of β meeting B . A space X is said to be *fully normal* iff every open cover of X has an open star-refinement. Prove that a regular space is fully normal iff it is paracompact. [Stone 1.]

3. CONVERGENCE

This section is devoted to a brief exposition of Moore-Smith convergence.

A binary relation $\geq$ in a set D is said to *direct* D iff D is nonempty and the following three conditions are satisfied:

- (DS1) If $a \in D$, then $a \geq a$.
- (DS2) If a, b, c are members of D such that $a \geq b$ and $b \geq c$, then $a \geq c$.
- (DS3) If a and b are members of D , then there exists a member $c \in D$ such that $c \geq a$ and $c \geq b$.

EXAMPLES. The set N of all natural numbers as well as the set R of all real numbers are directed by $\geq$ in the usual sense. Also, every local basis of a point in a space is directed by the inclusion $\subset$.

By a *directed set*, we mean a set D furnished with a binary relation $\geq$ which directs D . In particular, the set N of all natural numbers together with the usual relation $\geq$ is a directed set.

Let D be a given directed set and consider an arbitrary subset E of D . If there exists an element $d_0 \in D$ such that, for each $d \in D$, $d \geq d_0$ implies $d \in E$, then E is called a *residual subset* of D . If, for every $d \in D$, there exists an $e \in E$ such that $e \geq d$, then E is said to be a *cofinal subset* of D . It is easy to see that every residual subset of D is cofinal and that every cofinal subset of D is directed by $\geq$.

By a *net* in a space X , we mean a function

$$\phi : D \rightarrow X$$

from a directed set D into X . In particular, if the domain D is the set N of all natural numbers directed by $\geq$ in the usual sense, then ϕ is called a *sequence* in X .

Now, let $\phi : D \rightarrow X$ be a given net in a space X and let U be any subset of X . The net ϕ is said to be *in* U iff $\phi(D) \subset U$. The net ϕ is said to be *eventually in* U iff there exists a residual subset E of D such that $\phi(E) \subset U$. The net ϕ is said to be *frequently in* U iff there exists a cofinal subset C of D such that $\phi(C) \subset U$.

A net $\phi : D \rightarrow X$ in a space X is said to *converge* to a point $p \in X$ iff ϕ is eventually in every neighborhood of p . If X is indiscrete, then every net in X converges to every point of X . On the other hand, if X is discrete, then a net ϕ in X converges to a point $p \in X$ iff $\phi(d) = p$ for all $d \geq$ some $d_0 \in D$.

The convergence of nets in a space X characterizes the topology of the space X . Precisely, we have the following theorem.

THEOREM 3.1. *A set U in a space X is open iff no net in $X \setminus U$ can converge to a point of U .*

Proof: Necessity. Assume that U is open and that $\phi : D \rightarrow X$ is a net which converges to a point $p \in U$. Since U is a neighborhood of p , it follows from the definition of convergence that ϕ must be eventually in U . This implies that ϕ can never be in $X \setminus U$.

Sufficiency. Assume that U is not open. Then there exists a point $p \in U$ such that every neighborhood of p meets $X \setminus U$. Let D be a local basis at the point p in X . Then D is directed by the inclusion $\subset$. For each neighborhood $N \in D$, pick a point $\phi(N)$ in $N \cap (X \setminus U)$. The assignment $N \rightarrow \phi(N)$ defines a net $\phi : D \rightarrow X$ which converges to p and is in $X \setminus U$. Hence the condition cannot hold. $\parallel$

COROLLARY 3.2. *A set U in a space X satisfying the first axiom of countability is open iff no sequence in $X \setminus U$ can converge to a point of U .*

This corollary follows from the proof of (3.1) and the following lemma.

LEMMA 3.3. *Every point p of a space X satisfying the first axiom of countability has a local basis which consists of a decreasing sequence of open neighborhoods of p .*

Proof. Let $\beta = \{U_1, U_2, \dots, U_n, \dots\}$ be a countable local basis of X at the point p . For each $n = 1, 2, \dots$, let

$$V_n = U_1 \cap U_2 \cap \dots \cap U_n.$$

Then the sequence $\gamma = \{V_1, V_2, \dots, V_n, \dots\}$ is also a local basis of X at p . Since $V_n \supset V_{n+1}$ for each $n = 1, 2, \dots$, this completes the proof of (3.3). $\square$

Since, by (3.1), the notion of convergence determines the topology of the space, it should be able to characterize the features defined by the topology of a space, say for example, the closure and the accumulation points of a set E in a space X .

PROPOSITION 3.4. *A point p of a space X belongs to the closure $\text{Cl}(E)$ of a set E in X iff there exists a net ϕ in E which converges to p .*

Proof: Necessity. Let D be any local basis at the point p . Then D is directed by the inclusion $\subset$. For each $U \in D$, the intersection $E \cap U$ is nonempty since p is a point of $\text{Cl}(E)$. Choose a point $\phi(U)$ in $E \cap U$ for each $U \in D$. The assignment $U \rightarrow \phi(U)$ defines a net ϕ in E which obviously converges to p .

Sufficiency. Let $\phi : D \rightarrow E$ be an arbitrary net in E which converges to a point p of X . Let U be an arbitrary neighborhood of p . Since ϕ converges to p there exists a residual subset R of D such that $\phi(R) \subset U$. In particular, $E \cap U \neq \emptyset$. Hence, $p \in \text{Cl}(E)$. $\square$

PROPOSITION 3.5. *A point p of a space X is an accumulation point of a set E in X iff there exists a net ϕ in $E \setminus \{p\}$ which converges to p .*

Proof: Necessity. Let p be any accumulation point of E and let D be a local basis at p . Then D is directed by the inclusion $\subset$. For each $U \in D$, the intersection $U \cap (E \setminus \{p\})$ is nonempty and hence we may pick a point $\phi(U)$ in $U \cap (E \setminus \{p\})$. The assignment $U \rightarrow \phi(U)$ defines a net ϕ in $E \setminus \{p\}$ which obviously converges to p .

Sufficiency. Let $\phi : D \rightarrow E \setminus \{p\}$ be a net in $E \setminus \{p\}$ which converges to p . Let U be any neighborhood of p . Since ϕ converges to p ,

there exists a residual subset R of D such that $\phi(R) \subset U$. In particular, $U \cap (E \setminus \{p\}) \neq \emptyset$. Hence, p is an accumulation point of E in X . $\parallel$

As we have seen above, a net in a space may converge to several different points. However, in a Hausdorff space, convergence is unique. Precisely, we have the following theorem.

THEOREM 3.6. *A space X is a Hausdorff space iff every net in X can converge to at most one point.*

Proof: Necessity. Let X be a Hausdorff space and a, b be two distinct points of X . There are open sets U and V of X such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$. By the definition of residual subsets of a directed set, it is clear that no net can be eventually in both U and V . Hence, no net in X can converge to both a and b .

Sufficiency. Assume that a given space X is not a Hausdorff space. Then, there exist two distinct points a and b in X such that every neighborhood of a meets every neighborhood of b . Let A be a local basis at a and let B be a local basis at b . Both A and B are directed by the inclusion $\subset$. Consider the cartesian product

$$D = A \times B.$$

Introduce a binary relation $\geq$ as follows. Let (U_1, V_1) and (U_2, V_2) be two elements in D . We define

$$(U_1, V_1) \geq (U_2, V_2)$$

iff $U_1 \subset U_2$ and $V_1 \subset V_2$. Clearly, D is directed by $\geq$. For each $(U, V) \in D$, the intersection $U \cap V \neq \emptyset$ and hence we may pick a point $\phi(U, V) \in U \cap V$. This assignment $(U, V) \rightarrow \phi(U, V)$ defines a net $\phi : D \rightarrow X$ which converges to both a and b . $\parallel$

If X is a Hausdorff space and a net $\phi : D \rightarrow X$ converges to a point $p \in X$, then the point p is said to be *the limit* of the net ϕ , in symbols,

$$\lim(\phi) = p.$$

A function $\lambda : E \rightarrow D$ from a directed set E into a directed set D is said to be *cofinal* iff, for every residual subset Q of D , there exists a residual subset R of E such that $\lambda(R) \subset Q$; in other words, for every $d \in D$, there exists an $e \in E$ such that $\lambda(x) \geq d$ for every $x \geq e$ in E .

By a *subnet* of a net $\phi : D \rightarrow X$, we mean a net $\psi : E \rightarrow X$ such that there exists a cofinal function $\lambda : E \rightarrow D$ with

$$\psi = \phi \circ \lambda.$$

In particular, if E is a cofinal subset of D and is directed by the same ordering in D , then the inclusion function $\lambda : E \subset D$ and the restriction

$$\phi \upharpoonright E = \phi \circ \lambda$$

is a subnet of ϕ .

PROPOSITION 3.7. *If $\phi : D \rightarrow X$ is a net in a space X and if $\mathcal{F}$ is a family of sets in X such that the intersection of any two members of $\mathcal{F}$ contains a member of $\mathcal{F}$ and such that the net ϕ is frequently in each member of $\mathcal{F}$, then there exists a subnet of ϕ which is eventually in each member of $\mathcal{F}$.*

Proof. Since the intersection of any two members of $\mathcal{F}$ contains a member of $\mathcal{F}$, $\mathcal{F}$ is directed by the inclusion $\subset$. Consider the subset E of the cartesian product $D \times \mathcal{F}$ defined by

$$E = \{(d, U) \in D \times \mathcal{F} \mid \phi(d) \in U\}.$$

Then E is directed by the binary order $\geq$ defined as follows. Let (d_1, U_1) and (d_2, U_2) be any two elements in E . Then $(d_1, U_1) \geq (d_2, U_2)$ iff $d_1 \geq d_2$ and $U_1 \subset U_2$.

Define a function $\lambda : E \rightarrow D$ by taking $\lambda(d, U) = d$ for each element (d, U) in E . Obviously, λ is cofinal and hence $\psi = \phi \circ \lambda : E \rightarrow X$ is a subnet of ϕ .

Finally, let U_0 be an arbitrary member of $\mathcal{F}$. Since ϕ is frequently in U_0 , there exists a member d_0 of D such that $\phi(d_0) \in U_0$. Hence $e_0 = (d_0, U_0)$ is a member of E . Now, let $e = (d, U)$ be an arbitrary element of E such that $e \geq e_0$, in other words, $d \geq d_0$ and $U \subset U_0$. Then, we have

$$\psi(e) = \phi[\lambda(d, U)] = \phi(d) \in U \subset U_0.$$

Hence ψ is eventually in U_0 . ||

Now, let us consider a given net

$$\phi : D \rightarrow X$$

in a space X . A point $p \in X$ is called a *cluster point* of the net ϕ iff ϕ is frequently in every neighborhood of p .

PROPOSITION 3.8. *A point p in a space X is a cluster point of a net $\phi : D \rightarrow X$ iff there exists a subnet $\psi : E \rightarrow X$ which converges to p .*

Proof: Necessity. Let p be a cluster point of a net $\phi : D \rightarrow X$ and let $\mathcal{F}$ be a local basis at the point p . Then the intersection of any two members of $\mathcal{F}$ contains a member of $\mathcal{F}$, and ϕ is frequently in each member of $\mathcal{F}$. Hence, by (3.7), there exists a subnet $\psi : E \rightarrow X$ of ϕ

which is eventually in each member of $\mathcal{F}$. Since $\mathcal{F}$ is a local basis at p , this implies that ψ converges to p .

Sufficiency. Assume that the net ϕ has a subnet ψ which converges to p . To prove that p is a cluster point of ϕ , let U be a neighborhood of p and d_0 any element in D . It suffices to find an element $d \in D$ such that $d \geq d_0$ and $\phi(d) \in U$. Since ψ is a subnet of ϕ , there exists a cofinal function $\lambda: E \rightarrow D$ such that $\psi = \phi \circ \lambda$. Since λ is cofinal, there is an element $e_0 \in E$ such that $\lambda(e) \geq d_0$ for every $e \geq e_0$. Since ψ converges to p , there exists an element $e \geq e_0$ with $\psi(e) \in U$. Now, let $d = \lambda(e)$. Then we have $d \geq d_0$ and $\phi(d) = \phi[\lambda(e)] = \psi(e) \in U$. $\parallel$

The compactness of a space X can be characterized by the convergence of nets as follows.

THEOREM 3.9. *A space X is compact iff every net in X has a subnet which converges to a point of X .*

Proof. According to (3.8), it suffices to prove that a space is compact iff every net in X has a cluster point.

Necessity. Let $\phi: D \rightarrow X$ be a net in a compact space X . For each $a \in D$, let M_a denote the subset of X which consists of the points $\phi(d)$ for all $d \geq a$ in D . Since D is directed by $\geq$, it follows that $\{M_a \mid a \in D\}$ has the finite intersection property and hence the family $\{\text{Cl}(M_a) \mid a \in D\}$ also has the finite intersection property. Since X is compact, it follows from (2.1) that there is a point $p \in X$ which belongs to $\text{Cl}(M_a)$ for every $a \in D$. We are going to prove that p is cluster point of ϕ . For this purpose, let U be a neighborhood of p and $a \in D$. Since $p \in \text{Cl}(M_a)$, it follows that $M_a \cap U \neq \emptyset$. Hence there exists an element $d \in D$ such that $d \geq a$ and $\phi(d) \in U$. This implies that ϕ is frequently in any neighborhood of p and consequently p is a cluster point of ϕ .

Sufficiency. Assume that every net in a space X has a cluster point. We are going to prove that X is compact. For this purpose, consider a family $\mathcal{F}$ of closed sets in X with the finite intersection property. Let $\mathcal{G}$ denote the family of all finite intersections of members of $\mathcal{F}$; then $\mathcal{G}$ also has the finite intersection property. Since $\mathcal{F} \subset \mathcal{G}$, it suffices to prove that the intersection of all members of $\mathcal{G}$ is nonempty. Since the intersection of any two members of $\mathcal{G}$ is a member of $\mathcal{G}$, $\mathcal{G}$ is directed by the inclusion $\subset$. In each member B of $\mathcal{G}$, choose a point $\phi(B) \in B$. This assignment $B \rightarrow \phi(B)$ defines a net $\phi: \mathcal{G} \rightarrow X$. By our hypothesis, ϕ has a cluster point $p \in X$. It remains to prove that the point p belongs to every member B of $\mathcal{G}$. Let $B \in \mathcal{G}$ and let C be any member of $\mathcal{G}$

with $C \subset B$. Then $\phi(C) \in C \subset B$; hence, ϕ is eventually in the closed set B . By (3.1) and (3.8), p must belong to B . $\square$

On the other hand, the continuity of a map $f: X \rightarrow Y$ can also be characterized by the convergence of nets as follows.

THEOREM 3.10. *A function $f: X \rightarrow Y$ from a space X into a space Y is continuous at a point $p \in X$ iff, for each net $\phi: D \rightarrow X$ which converges to p , the composition $\psi = f \circ \phi: D \rightarrow Y$ converges to $f(p)$.*

Proof: Necessity. Assume that $f: X \rightarrow Y$ is continuous at p and let V be any neighborhood of $f(p)$ in Y . By definition of continuity at p , there is a neighborhood U of p in X such that $f(U) \subset V$. Since ϕ converges to p , there exists a residual subset R of D such that $\phi(R) \subset U$. Then we have

$$\psi(R) = f[\phi(R)] \subset f(U) \subset V.$$

Hence ψ converges to $f(p)$.

Sufficiency. Assume that f is not continuous at p . Then there exists an open neighborhood V of $f(p)$ in Y such that every neighborhood of p in X meets $f^{-1}(Y \setminus V)$. Let D be a local basis at p directed by the inclusion $\subset$. In each $U \in D$, pick a point $\phi(U)$ in $U \cap f^{-1}(Y \setminus V)$. The assignment $U \rightarrow \phi(U)$ defines a net $\phi: D \rightarrow X$ which converges to p . Now, the composition $\psi = f \circ \phi: D \rightarrow Y$ is a net in $Y \setminus V$. Since V is open and $f(p) \in V$, ψ can never converge to $f(p)$. $\square$

EXERCISES

- 3A. Show that, for spaces satisfying the first axiom of countability, one can replace “net” by “sequence” in the assertions in this section except Theorem 3.9.
- 3B. Prove that a net in a topological product converges to a point p iff its projection in each coordinate space converges to the projection of the point p .
- 3C. Prove that a Lindelöf T_1 -space X is compact if every sequence in X has a cluster point.

4. CONNECTEDNESS

By a *separation* of a space X , we mean a pair of open sets U and V of X such that

$$U \cup V = X, \quad U \cap V = \square;$$

in symbols, $X = U \mid V$. The separation $U \mid V$ of X is said to be *trivial* iff one of the open sets U, V is empty; otherwise, it is said to be *nontrivial*. If $U \mid V$ is a separation of a space X , then each of the sets U and V is both open and closed.

A space X is said to be *connected* iff it has no nontrivial separation; otherwise, it is said to be *disconnected*. In other words, a space X is connected iff it contains no nonempty proper subset which is both open and closed. For examples, every indiscrete space is connected and every discrete space containing more than one point is disconnected.

A set E in a space X is said to be *connected* iff, with the relative topology, the subspace E is connected. One can easily verify that a separation $U \mid V$ of the subspace E is a pair of subsets U and V of the space X such that

$$U \cup V = E, \quad \text{Cl } U \cap V = \emptyset = U \cap \text{Cl } V,$$

where the closure operation Cl is taken in the space X .

PROPOSITION 4.1. *The real line R is connected.*

Proof. Assume that $U \mid V$ were a nontrivial separation of R . Let $a \in U$ and $b \in V$. Without loss of generality, we may assume that $a < b$. Consider the closed interval $[a, b]$ of real numbers and the subsets

$$A = U \cap [a, b], \quad B = V \cap [a, b].$$

As closed subsets of a compact set $[a, b]$, A and B are compact. Hence, the topological product $A \times B$ is compact. Define a continuous function $d : A \times B \rightarrow R$ by setting

$$d(x, y) = |x - y|, \quad (x \in A, y \in B).$$

The image $d(A \times B)$ is compact and therefore closed in R . Hence, $d(A \times B)$ contains its greatest lower bound δ ; that is, there exist $u \in \underline{A}$ and $v \in \underline{B}$ such that $d(u, v) = \delta > 0$. Consider the point $w = \frac{1}{2}(u + v)$. Then

$$d(u, w) = \frac{1}{2}\delta, \quad d(w, v) = \frac{1}{2}\delta.$$

Hence w can belong to neither U nor V . This contradiction proves (4.1). $\parallel$

COROLLARY 4.2. *Every open interval of the real line is connected.*

PROPOSITION 4.3. *If C is a connected set in a space X and E is a set in X such that $C \subset E \subset \text{Cl } (C)$, then E is connected.*

Proof. Let $U \mid V$ be an arbitrary separation of the subspace E of

X . Then $(U \cap C) \mid (V \cap C)$ is a separation of the subspace C . Since C is connected, one of the sets $U \cap C$ and $V \cap C$ must be empty. Therefore without loss of generality, we may assume that $V \cap C = \square$. This implies that $C \subset U$ and hence $E \subset \text{Cl}(C) \subset \text{Cl}(U)$. Thus, we have

$$E \subset E \cap \text{Cl}(U) = U$$

since U is closed in the subspace E . It follows that $V = \square$ and hence the separation $U \mid V$ of E is trivial. This implies that E is connected. $\parallel$

COROLLARY 4.4. *The closed unit interval I is connected.*

PROPOSITION 4.5. *Every continuous image of a connected set is connected.*

Proof. It suffices to prove that, if $f: X \rightarrow Y$ is a surjective map of a connected space X onto a space Y , then Y is also connected. For this purpose, let $U \mid V$ be an arbitrary separation of the space Y . Let $M = f^{-1}(U)$ and $N = f^{-1}(V)$. Then, M and N are open sets of X satisfying $X = M \cup N$ and $M \cap N = \square$. Hence $M \mid N$ is a separation of X . Since X is connected, one of the sets M and N must be empty. Without loss of generality, we may assume that $N = \square$. Since f is surjective and $N = f^{-1}(V)$, this implies that $V = \square$. Hence, Y is connected. $\parallel$

PROPOSITION 4.6. *If two connected sets A and B in a space X have a common point p , then $A \cup B$ is connected.*

Proof. Let $U \mid V$ be an arbitrary separation of the subspace $A \cup B$ with $p \in U$. Then it follows that $(A \cap U) \mid (A \cap V)$ is a separation of the subspace A . Since A is connected, one of the sets $A \cap U$ and $A \cap V$ must be empty. Since $p \in A \cap U$, we have $A \cap V = \square$. Hence, $A \subset U$. Similarly, one can prove $B \subset U$. Thus, $A \cup B \subset U$ and consequently $V = \square$. $\parallel$

Two points a and b of a space X are said to be *connected in X* iff there exists a connected set in X which contains both a and b .

PROPOSITION 4.7. *If every two points of a space X are connected in X , then X is connected.*

Proof. Assume that there were a nontrivial separation $U \mid V$ of the space X . Let $a \in U$ and $b \in V$. By hypothesis, there exists a connected set C in X which contains both a and b . Then $(U \cap C) \mid (V \cap C)$ is a nontrivial separation of C . This contradicts the connectedness of C and proves (4.7). $\parallel$

COROLLARY 4.8. *The union of any family of connected sets with a common point is connected.*

Define a binary relation $\sim$ in any given space X as follows: for any two points a and b of X , $a \sim b$ iff a and b are connected in X . By the aid of (4.6), one can easily verify that $\sim$ is an equivalence relation in X . Hence, the points of X are divided into disjoint equivalence classes called the *components* of the space X . We will denote by $C(p)$ the component of X which contains the point $p \in X$.

PROPOSITION 4.9. *For each point p in a space X , the component $C(p)$ of X is the largest connected set in X which contains the point p .*

Proof. Let K be any connected set in X which contains p and let q be an arbitrary point in K . By definition, we have $p \sim q$ and hence $q \in C(p)$. This implies $K \subset C(p)$.

It remains to prove the connectedness of $C(p)$. For this purpose, let r denote an arbitrary point in $C(p)$. Since $p \sim r$, there exists a connected set K_r of X which contains both p and r . By the first half of the proof, $K_r \subset C(p)$. Hence, $C(p)$ is the union of the family $\{K_r \mid r \in C(p)\}$ of connected sets with a common point p . It follows from (4.8) that $C(p)$ is connected. $\parallel$

COROLLARY 4.10. *For each point p in a space X , the component $C(p)$ of X is a closed set of X .*

Proof. By (4.9), $C(p)$ is the largest connected set of X containing the point p . By (4.3), the closure $\text{Cl}[C(p)]$ of $C(p)$ is connected. Hence $\text{Cl}[C(p)] = C(p)$ and $C(p)$ is closed. $\parallel$

PROPOSITION 4.11. *The topological product of an arbitrary family of connected spaces is connected.*

Proof. First, let us prove that the topological product $X \times Y$ of two connected spaces X and Y is connected. For this purpose, let (a, b) and (c, d) be any two points of $X \times Y$ and consider the point (a, d) . Since the connected set $a \times Y$ contains both (a, b) and (a, d) , we have $(a, b) \sim (a, d)$. On the other hand, since the connected set $X \times d$ contains both (a, d) and (c, d) , we have $(a, d) \sim (c, d)$. Hence $(a, b) \sim (c, d)$ and $X \times Y$ is connected according to (4.7). It follows from a finite induction that the topological product of a finite family of connected spaces is connected.

Now, consider an arbitrary family

$$\mathcal{F} = \{X_\mu \mid \mu \in M\}$$

of connected spaces. Let X denote the topological product of this

family and let p be a given point in X . Consider the component $C(p)$ of X which contains the point p . Let B denote an arbitrary basic open set of the topological product X . Then there are a finite number of elements of M , say, $\mu_1, \dots, \mu_n$, and open sets

$$U_i \subset X_{\mu_i}, \quad (i = 1, \dots, n),$$

such that B consists of the points $x \in X$ satisfying $x(\mu_i) \in U_i$ for each $i = 1, \dots, n$. In each U_i , ($i = 1, \dots, n$), pick a point a_i . Let q denote the point of X defined by

$$q(\mu) = \begin{cases} a_i, & (\text{if } \mu = \mu_i \text{ for some } i), \\ p(\mu), & (\text{if } \mu \neq \mu_i \text{ for all } i). \end{cases}$$

Then $q \in B$.

Consider the subset K of X which consists of all the points $x \in X$ such that $x(\mu) = p(\mu)$ for all $\mu \in M$ different from $\mu_1, \dots, \mu_n$. Since K is homeomorphic to

$$X_{\mu_1} \times \dots \times X_{\mu_n},$$

K is connected. Since K contains both p and q , it follows that $q \in C(p)$. Hence,

$$B \cap C(p) \neq \emptyset.$$

Since B is an arbitrary basic open set of X , this implies that $C(p)$ is dense in X . Finally, since $C(p)$ is closed in X , we have $C(p) = X$. Hence, X is connected. $\square$

COROLLARY 4.12. *For every $n \geq 1$, the n -dimensional Euclidean space R^n , the unit n -cube I^n , and the n -sphere S^n are connected.*

Proof. That R^n and I^n are connected follows from (4.11) together with (4.1) and (4.4) respectively. To prove that S^n is connected, let $p \in S^n$. Since the subspace $S^n \setminus p$ is homeomorphic to R^n , it is connected. Hence, by (4.3), $\text{Cl}(S^n \setminus p) = S^n$ is also connected. $\square$

Now, denote by $\Gamma(X)$ the set of all components of a space X . Consider an arbitrarily given map

$$f : X \rightarrow Y.$$

For each component α of X , the image $f(\alpha)$ is connected and hence is contained in a component β of Y . The assignment $\alpha \rightarrow \beta$ defines a function

$$\Gamma(f) : \Gamma(X) \rightarrow \Gamma(Y).$$

LEMMA 4.13. *If two maps $f, g : X \rightarrow Y$ are homotopic, then*

$$\Gamma(f) = \Gamma(g).$$

Proof. By definition of homotopic maps, there exists a map

$$H : X \times I \rightarrow Y$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for every $x \in X$. Let α be an arbitrary component of X . Then, as product of connected sets, $\alpha \times I$ is connected. It follows that $H(\alpha \times I)$ is connected and hence is contained in a component β of Y . Since $f(\alpha)$ and $g(\alpha)$ are both contained in $H(\alpha \times I)$, we have

$$[\Gamma(f)](\alpha) = \beta = [\Gamma(g)](\alpha).$$

Since α is an arbitrary element of $\Gamma(X)$, this proves $\Gamma(f) = \Gamma(g)$. ||

PROPOSITION 4.14. *If $f : X \rightarrow Y$ is a homotopy equivalence, then $\Gamma(f)$ is bijective.*

Proof. Let $g : Y \rightarrow X$ be a map such that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity maps i_X and i_Y on the spaces X and Y respectively. Hence, by (4.13),

$$\begin{aligned}\Gamma(g) \circ \Gamma(f) &= \Gamma(g \circ f) = \Gamma(i_X), \\ \Gamma(f) \circ \Gamma(g) &= \Gamma(f \circ g) = \Gamma(i_Y)\end{aligned}$$

are identity functions on the sets $\Gamma(X)$ and $\Gamma(Y)$ respectively. This implies that $\Gamma(f)$ is bijective and $\Gamma(g)$ is the inverse of $\Gamma(f)$. ||

COROLLARY 4.15. *Connectedness is a homotopy property of spaces.*

Now, let us turn to an application of connectedness. For this purpose, let X denote the closed interval $[a, b]$ as a subspace of the real line R and let A denote the subspace of X which consists of the two end points a and b .

LEMMA 4.16. *The subspace A is not a retract of X .*

Proof. Assume that there were a retraction $r : X \supset A$. Then r is surjective. Since X is connected, A should be also connected. However, A is obviously disconnected since the open sets $\{a\}$ and $\{b\}$ of the discrete space A form a nontrivial separation of A . This contradiction proves (4.16). ||

PROPOSITION 4.17 (FIXED POINT THEOREM). *Every map*

$$f : [a, b] \rightarrow [a, b]$$

of the closed interval $[a, b]$ into itself has at least one fixed point, that is to say, there exists a real number $x \in [a, b]$ such that $f(x) = x$.

Proof. Without loss of generality, we may assume that $a = -1$ and $b = 1$.

To prove (4.17) by contradiction, let us assume $f(x) \neq x$ for every real number $x \in [-1, 1]$. Define a continuous function r on $[-1, 1]$ by taking

$$r(x) = \frac{x - f(x)}{|x - f(x)|}, \quad (-1 \leq x \leq 1),$$

where $|x - f(x)|$ denotes the absolute value of the real number $x - f(x)$. The continuity of r follows from that of f and the assumption that $f(x) \neq x$ for all $x \in [-1, 1]$. One can easily verify that r is a retraction of $X = [-1, 1]$ onto its boundary 0-sphere $A = \{-1, 1\}$. This contradicts (4.16) and proves (4.17). ||

The remainder of this section is devoted to local connectedness.

A space X is said to be *locally connected at a point* $p \in X$ iff, for every neighborhood U of p , there exists a neighborhood $V \subset U$ of p such that any two points of V are connected in U .

PROPOSITION 4.18. *A space X is locally connected at a point p iff every neighborhood of p contains a connected neighborhood of p .*

Proof: Necessity. Assume X to be locally connected at p and let U be an arbitrary neighborhood of p . By definition, there exists a neighborhood $V \subset U$ of p such that any two points of V are connected in U . Let q be an arbitrary point of V . Then there exists a connected set C_q in U which contains both p and q . Let

$$C = \bigcup \{C_q \mid q \in V\} \subset U.$$

Since $V \subset C$, C is a neighborhood of p . By (4.8), C is connected. Therefore, C is a connected neighborhood of p contained in U .

Sufficiency. Assume that the condition holds. Let U be an arbitrary neighborhood of p in X . Then U contains a connected neighborhood V of p . Since V is connected, any two points of V are connected in V and hence also in U . Hence, X is locally connected at p . ||

COROLLARY 4.19. *If a space X is locally connected at a point $p \in X$, then p is an interior point of the component $C(p)$ of X .*

Proof. By (4.18), p has a connected neighborhood V . By (4.9), $V \subset C(p)$. Hence, p is an interior point of $C(p)$. ||

A space X is said to be *locally connected* iff it is locally connected at each of its points. The following proposition is an immediate consequence of the definition.

PROPOSITION 4.20. *Every open subspace of a locally connected space is locally connected.*

On the other hand, the following proposition is an immediate consequence of (4.19).

PROPOSITION 4.21. *For each point p of a locally connected space X , the component $C(p)$ of X is an open set of X .*

PROPOSITION 4.22. *For a given space X , the following statements are equivalent:*

- (a) *X is locally connected.*
- (b) *The components of every open subspace of X are open.*
- (c) *The connected open sets of X form a basis of the topology of X .*

Proof. (a) $\Rightarrow$ (b). Let X be a locally connected space and let U be an open subspace of X . By (4.20), U is a locally connected space. By (4.21), the components of U are open sets of U . Since U is open, these components are also open in X .

(b) $\Rightarrow$ (c). Assume that (b) holds and let U be any open set of X . By (b), the components of U are open in X . By (4.9), the components of U are connected. Hence, U is the union of a collection of connected open sets of X . This proves (c).

(c) $\Rightarrow$ (a). Assume that (c) holds and let U be any open neighborhood of an arbitrary point p in X . By (c), U is the union of a collection of connected open sets. Hence, there exists a connected open set V such that $p \in V \subset U$. By (4.18), X is locally connected at p . Since p is arbitrary, this implies that X is locally connected. ||

EXERCISES

- 4A. Prove that, if a connected set C in a space X meets a set $E \subset X$ and its complement $X \setminus E$, then C meets the boundary ∂E of E .
- 4B. Prove that, if $\{C_\mu \mid \mu \in M\}$ is a family of connected sets in a space X such that each C_μ meets a given connected set C in X , then the union of C and all C_μ , $\mu \in M$, is connected.
- 4C. Prove that, if $f: X \rightarrow Y$ is a closed map of a locally connected space X onto a space Y , then Y is locally connected. Hence every continuous image of the unit interval I in a Hausdorff space is locally connected.

5. PATHWISE CONNECTEDNESS

By a *path* in a space X , we mean a map

$$\sigma : I \rightarrow X$$

of the unit interval $I = [0, 1]$ into the space X . The points $\sigma(0)$ and $\sigma(1)$ in X are called the *initial point* and the *final point* of the path σ respectively.

In a given space X , let us define an equivalence relation $\sim$ as follows: for any two points a and b in X ,

$$a \sim b$$

iff there exists a path $\sigma : I \rightarrow X$ such that $\sigma(0) = a$ and $\sigma(1) = b$. This equivalence relation $\sim$ divides X into disjoint equivalence classes called the *path-components* of the space X . If the space X has no more than one path-component, it is said to be *pathwise connected*. In other words, a space X is pathwise connected iff, for any two points a and b in X , there exists a path $\sigma : I \rightarrow X$ such that $\sigma(0) = a$ and $\sigma(1) = b$. A set E in a space X is said to be *pathwise connected* iff, with the relative topology, the subspace E is pathwise connected.

PROPOSITION 5.1. *Every pathwise connected space is connected.*

Proof. Let X be an arbitrary pathwise connected space. Since (5.1) is trivial in case $X = \square$, we assume that X is nonempty. Pick a point $x_0 \in X$.

Since X is pathwise connected, we may choose for each point $x \in X$ a path

$$\sigma_x : I \rightarrow X$$

such that $\sigma_x(0) = x_0$ and $\sigma_x(1) = x$. As continuous image of a connected space I , $\sigma_x(I)$ is a connected set for every $x \in X$. Since $x_0 \in \sigma_x(I)$ and since

$$X = \bigcup_{x \in X} \sigma_x(I),$$

it follows from (4.8) that X is connected. $\parallel$

Thus, pathwise connectedness is a stronger form of connectedness. On the other hand, we have the following

PROPOSITION 5.2. *Every contractible space is pathwise connected.*

Proof. Let X be an arbitrary contractible space. Then there exists a contraction

$$\{h_t : X \rightarrow X \mid t \in I\}.$$

Let a and b be any two points in X . Define a map $\sigma : I \rightarrow X$ by taking

$$\sigma(t) = \begin{cases} h_{2t}(a), & (\text{if } 0 \leq t \leq \frac{1}{2}), \\ h_{2-2t}(b), & (\text{if } \frac{1}{2} \leq t \leq 1). \end{cases}$$

This definition is justified by the fact that

$$h_1(a) = h_1(X) = h_1(b)$$

since $\{h_t\}$ is a contraction. Since

$$\sigma(0) = h_0(a) = a, \quad \sigma(1) = h_0(b) = b,$$

the space X is pathwise connected. $\parallel$

In particular, the real line R , the unit interval I , the unit n -simplex Δ^n , and the cone $\text{Con}(X)$ over a space X are pathwise connected.

PROPOSITION 5.3. *Every continuous image of a pathwise connected set is pathwise connected.*

Proof. It suffices to prove that, if $f : X \rightarrow Y$ is a surjective map of a pathwise connected space X onto a space Y , then Y is also pathwise connected. For this purpose, let u and v be any two points in Y . Since the map $f : X \rightarrow Y$ is surjective, there exist two points a and b in X such that $f(a) = u$ and $f(b) = v$. Since X is pathwise connected, there exists a path

$$\sigma : I \rightarrow X$$

such that $\sigma(0) = a$ and $\sigma(1) = b$. Consider the composed map

$$\tau = f \circ \sigma : I \rightarrow Y.$$

Then τ is a path in Y with

$$\tau(0) = f(a) = u, \quad \tau(1) = f(b) = v.$$

Hence Y is pathwise connected. $\parallel$

PROPOSITION 5.4. *The topological product of an arbitrary family of pathwise connected spaces is pathwise connected.*

Proof. Consider an arbitrary family

$$\mathcal{F} = \{X_\mu \mid \mu \in M\}$$

of pathwise connected spaces. Let Φ denote the topological product of this family and let f and g be any two points in Φ .

For each $\mu \in M$, $f(\mu)$ and $g(\mu)$ are points of a pathwise connected space X_μ . Hence, there exists a path

$$\sigma_\mu : I \rightarrow X_\mu$$

such that $\sigma_\mu(0) = f(\mu)$ and $\sigma_\mu(1) = g(\mu)$.

Then, the restricted cartesian product

$$\sigma : I \rightarrow \Phi$$

of the family $\{\sigma_\mu : I \rightarrow X_\mu \mid \mu \in M\}$ is a path in Φ with $\sigma(0) = f$ and $\sigma(1) = g$. Hence Φ is pathwise connected. $\parallel$

In particular, the topological powers R^M and I^M for any set M are pathwise connected.

Now, let us study the path-components of a given space X . We will denote by $c(p)$ the path-component of X which contains the point $p \in X$.

PROPOSITION 5.5. *For each point p in a space X , the path-component $c(p)$ of X is the largest pathwise connected set in X which contains the point p .*

Proof. Let K be any pathwise connected set in X which contains the point p , and let q be an arbitrary point in K . Since K is pathwise connected, there exists a path $\sigma : I \rightarrow K$ with $\sigma(0) = p$ and $\sigma(1) = q$. Then the composed map

$$\tau := i \circ \sigma : I \rightarrow X$$

of $\sigma : I \rightarrow K$ and the inclusion map $i : K \subset X$ is a path in X with $\tau(0) = p$ and $\tau(1) = q$. Hence, $p \sim q$ and $q \in c(p)$. This proves that $K \subset c(p)$.

It remains to prove the pathwise connectedness of $c(p)$. For this purpose, let r denote an arbitrary point in $c(p)$. Since $p \sim r$, there exists a path $\sigma : I \rightarrow X$ with $\sigma(0) = p$ and $\sigma(1) = r$. For the pathwise connectedness of $c(p)$, it suffices to prove that

$$\sigma(I) \subset c(p).$$

Let $s \in I$ be arbitrarily given. Define a path $\tau : I \rightarrow X$ by taking

$$\tau(t) = \sigma(st)$$

for every $t \in I$. Then we have $\tau(0) = p$ and $\tau(1) = \sigma(s)$. Hence $p \sim \sigma(s)$ and $\sigma(s) \in c(p)$. This proves $\sigma(I) \subset c(p)$. $\parallel$

Now, denote by $\pi_0(X)$ the set of all path-components of a space X . Consider an arbitrarily given map

$$f : X \rightarrow Y.$$

For each path-component α of X , the image $f(\alpha)$ is pathwise connected and hence is contained in a path-component β of Y . The assignment $\alpha \rightarrow \beta$ defines a function

$$\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y).$$

LEMMA 5.6. *If two maps $f, g : X \rightarrow Y$ are homotopic, then*

$$\pi_0(f) = \pi_0(g).$$

PROPOSITION 5.7. *If $f : X \rightarrow Y$ is a homotopy equivalence, then $\pi_0(f)$ is bijective.*

COROLLARY 5.8. *Pathwise connectedness is a homotopy property of spaces.*

The proof of the preceding assertions (5.6)–(5.8) is similar to that of (4.13)–(4.15) and hence is left to the student.

A space X is said to be *locally pathwise connected at a point* $p \in X$ iff, for every neighborhood U of p , there exists a neighborhood $V \subset U$ of p such that every point $q \in V$ is the final point of a path

$$\sigma_q : I \rightarrow U$$

with initial point $\sigma_q(0) = p$.

PROPOSITION 5.9. *A space X is locally pathwise connected at a point p iff every neighborhood of p contains a pathwise connected neighborhood of p .*

Proof: Necessity. Assume X to be locally pathwise connected at p and let U be an arbitrary neighborhood of p . By definition, there exists a neighborhood $V \subset U$ of p such that, for each point $q \in V$, there exists a path

$$\sigma_q : I \rightarrow U$$

with $\sigma_q(0) = p$ and $\sigma_q(1) = q$. Hence, V is contained in the path-component $c_U(p)$ of U which contains p . By (5.5), it follows that $c_U(p)$ is a pathwise connected neighborhood of p contained in U .

Sufficiency. Assume that the condition holds. Let U be an arbitrary neighborhood of p in X . Then U contains a pathwise connected neighborhood V of p . Let q be an arbitrary point of V . Since V is pathwise connected, there is a path

$$\tau_q : I \rightarrow V$$

with $\tau_q(0) = p$ and $\tau_q(1) = q$. Then the composed map

$$\sigma_q = i \circ \tau_q : I \rightarrow U$$

of τ_q and the inclusion map $i: V \subset U$ is a path in U with $\sigma_q(0) = p$ and $\sigma_q(1) = q$. This proves that X is locally pathwise connected at p . $\parallel$

COROLLARY 5.10. *If a space X is locally pathwise connected at a point $p \in X$, then p is an interior point of the path-component $c(p)$ of X .*

COROLLARY 5.11. *If a space X is locally pathwise connected at a point $p \in X$, then X is locally connected at the point p .*

A space X is said to be *locally contractible at a point* $p \in X$ iff, for every neighborhood U of p in X , there exists a neighborhood $V \subset U$ of p which is contractible in U .

PROPOSITION 5.12. *If a space X is locally contractible at a point $p \in X$, then X is locally pathwise connected at the point p .*

Proof. Let U be an arbitrary neighborhood of p in X . Since X is locally contractible at the point p , there exists a neighborhood $V \subset U$ of p which is contractible in U . Let

$$\{h_t: V \rightarrow U \mid t \in I\}$$

be a contraction of V in U ; then h_0 is the inclusion map and h_1 is a constant map. Let q be an arbitrary point in V and define a path $\sigma_q: I \rightarrow U$ by taking

$$\sigma_q(t) = \begin{cases} h_{2t}(p), & (\text{if } 0 \leq t \leq \frac{1}{2}), \\ h_{2-2t}(q), & (\text{if } \frac{1}{2} \leq t \leq 1). \end{cases}$$

Then $\sigma_q(0) = p$ and $\sigma_q(1) = q$. This proves that X is locally pathwise connected at the point p . $\parallel$

A space X is said to be *locally pathwise connected* iff it is locally pathwise connected at each of its points. Similarly, a space X is said to be *locally contractible* iff it is locally contractible at each of its points.

The following two propositions are immediate consequences of the definitions.

PROPOSITION 5.13. *Every open subspace of a locally pathwise connected space is locally pathwise connected.*

PROPOSITION 5.14. *Every open subspace of a locally contractible space is locally contractible.*

The following two propositions are immediate consequences of (5.11) and (5.12).

PROPOSITION 5.15. *Every locally pathwise connected space is locally connected.*

PROPOSITION 5.16. *Every locally contractible space is locally pathwise connected.*

THEOREM 5.17. *For each point p of a locally pathwise connected space X , the path-component $c(p)$ of X is both open and closed in X and hence $c(p)$ coincides with the component $C(p)$ of X .*

Proof. Let q be an arbitrary point in $c(p)$. Then $c(q) = c(p)$. Hence, q is an interior point of $c(p)$ by (5.10). This proves that $c(p)$ is open.

The union U of all path-components of X other than $c(p)$ is an open set of X . Since the path-components are disjoint, we have

$$c(p) = X \setminus U.$$

This proves that $c(p)$ is closed.

By (5.5) and (5.1), $c(p)$ is connected. It follows that $c(p)$ is the largest connected set in X containing the point p . Hence $c(p) = C(p)$. ||

COROLLARY 5.18. *If a space X is connected and locally pathwise connected, then X is pathwise connected.*

In the remainder of the present section, we will briefly investigate the classical notion of arcwise connectedness.

By an *arc* in a space X , we mean an imbedding

$$\alpha : I \rightarrow X$$

of the unit interval I into the space X . If we use arcs instead of paths, we can define the notions of arcwise connectedness and arc-components of spaces.

Since every arc in a space is also a path, the following proposition is trivial.

PROPOSITION 5.19. *Every arcwise connected space is pathwise connected.*

However, the converse of (5.19) is false. For example, let X denote the indiscrete space consisting of two points p and q . Then X is pathwise connected because p and q are connected by the path $\sigma : I \rightarrow X$ defined by

$$\sigma(t) = \begin{cases} p, & (\text{if } t = 0), \\ q, & (\text{if } t > 0). \end{cases}$$

Since X has only two points, there can be no arc in X and hence X is not arcwise connected.

Since X and the singleton space $Y = \{p\}$ are both indiscrete, they are homotopically equivalent by (II, Ex. 7F). Since Y is arcwise connected while X is not, we have the following

PROPOSITION 5.20. *Arcwise connectedness is not a homotopy property.*

Because of (5.8) and (5.2), we prefer pathwise connectedness to arcwise connectedness, although the two notions coincide for Hausdorff spaces. See [H-Y, pp. 115-130].

EXERCISES

- 5A. Prove that the union of any family of pathwise connected sets with a common point is pathwise connected. Hence, the one point union of a family of pathwise connected spaces is pathwise connected.
- 5B. The *suspension* $S(X)$ of a space X is defined to be the quotient space obtained by collapsing the subset X of $\text{Con}(X)$ to a point. Prove that $S(X)$ is pathwise connected. If X is homeomorphic to the unit n -sphere S^n , then the suspension $S(X)$ is homeomorphic to S^{n+1} . Hence, S^n is pathwise connected for every $n > 0$.
- 5C. Prove that the mapping cylinder $\text{Cyl}(f)$ of a map $f: X \rightarrow Y$ from a space X into a pathwise connected space Y is pathwise connected.
- 5D. Prove that the following statements are equivalent for a space X :
 - (a) X is locally pathwise connected.
 - (b) The path-components of every open subspace of X are open.
 - (c) The pathwise connected open sets of X form a basis of the topology of X .

6. IMBEDDING THEOREMS

In the present section, we are concerned with the problem of imbedding a given space in a cube I^M .

Let us consider a given family

$$\mathcal{F} = \{f_\mu: X \rightarrow Y_\mu \mid \mu \in M\}$$

of maps from a given space X into a space Y_μ which may be different for various indices $\mu \in M$. Let Y denote the topological product of the family $\{Y_\mu \mid \mu \in M\}$ of spaces and denote by

$$f: X \rightarrow Y$$

the (restricted) Cartesian product of the family $\mathcal{F}$ as defined in (I, § 3). Then we have $[f(x)](\mu) = f_\mu(x)$ for every $x \in X$ and every $\mu \in M$. By (II, 5.8), f is continuous.

The family $\mathcal{F}$ is said to be *able to distinguish points of X* iff, for any two distinct points a and b of X , there exists a map $f_\mu \in \mathcal{F}$ such that

$$f_\mu(a) \neq f_\mu(b).$$

The family $\mathcal{F}$ is said to be *able to distinguish points from closed sets of X* iff, for any closed set A in X and any point p in $X \setminus A$, there exists a map $f_\mu \in \mathcal{F}$ such that $f_\mu(p)$ is not in the closure of $f_\mu(A)$.

LEMMA 6.1. *If the family $\mathcal{F}$ can distinguish points of X , then the Cartesian product $f: X \rightarrow Y$ is injective.*

Proof. Let a and b be any two distinct points of X . Then, there exists a map $f_\mu \in \mathcal{F}$ such that $f_\mu(a) \neq f_\mu(b)$. This implies $f(a) \neq f(b)$. ||

LEMMA 6.2. *If the family $\mathcal{F}$ can distinguish points from closed sets of X , then the Cartesian product $f: X \rightarrow Y$ carries open sets of X onto those of the subspace $f(X)$ of Y .*

Proof. Let U be an arbitrary open set of X . We will prove that $f(U)$ is an open set of $f(X)$. For this purpose, let q be an arbitrary point of $f(U)$ and pick a point $p \in U$ such that $f(p) = q$. Since $X \setminus U$ is closed and $p \notin X \setminus U$, there exists a map $f_\mu \in \mathcal{F}$ such that $f_\mu(p)$ is not in the closure $\text{Cl}[f_\mu(X \setminus U)]$ of $f_\mu(X \setminus U)$. Consider the subbasic open set

$$V = \{y \in Y \mid y(\mu) \notin \text{Cl}[f_\mu(X \setminus U)]\}$$

of the product space Y . Since $V \cap f(X)$ is an open set of $f(X)$ satisfying

$$q = f(p) \in V \cap f(X) \subset f(U),$$

it follows that $f(U)$ is an open set of $f(X)$. ||

Combining (6.1) and (6.2), we obtain the following theorem.

THEOREM 6.3. *If the family $\mathcal{F}$ of maps can distinguish points of X and can also distinguish points from closed sets of X , then the Cartesian product $f: X \rightarrow Y$ of the family $\mathcal{F}$ is an imbedding.*

COROLLARY 6.4 (TYCHONOFF IMBEDDING THEOREM). *Every Tychonoff space X can be imbedded as a subspace of a cube.*

Proof. Let M denote the family of all continuous real functions $f_\mu: X \rightarrow I$ on the Tychonoff space X into the closed unit interval $I = [0, 1]$. By means of the complete regularity, one can easily verify that M can distinguish points from closed sets of X . On the other hand,

since every singleton set of X is closed, M can also distinguish points of X . Therefore, the Cartesian product

$$f : X \rightarrow I^M$$

is an imbedding. ||

As a compact Hausdorff space, I^M is a Tychonoff space and so is every subspace of I^M . Hence, we have the following theorem.

THEOREM 6.5. *A space X is a Tychonoff space iff X is homeomorphic to a subspace of a cube.*

By (2.7), every cube I^M is a normal space. It follows from (6.4) that a subspace of a normal space may fail to be normal.

We recall that the cube I^N , where N denotes the set of all natural numbers, is called the *Hilbert cube*.

THEOREM 6.6 (URYSOHN IMBEDDING THEOREM). *Every regular Fréchet space X which satisfies the second axiom of countability can be imbedded as a subspace of the Hilbert cube I^N .*

Proof. In view of (6.3), it suffices to show that there exists a countable family $\mathcal{F}$ of continuous functions from X into I which distinguishes points from closed sets of X . For this purpose, let $\mathcal{B}$ denote a countable basis for the topology of X . Consider the Cartesian product $\mathcal{B} \times \mathcal{B}$ and its subcollection M which consists of the pairs (U, V) of basic open sets such that

$$\text{Cl}(U) \subset V.$$

Then, M is countable.

By Exercise 1E, X is a normal space. Hence, it follows from Urysohn's lemma (1.6) that, for each $(U, V) \in M$, there exists a continuous real function $f_{(U, V)} : X \rightarrow I$ which carries $\text{Cl}(U)$ into 0 and $X \setminus V$ into 1. Thus, we obtain a countable family

$$\mathcal{F} = \{f_{(U, V)} : X \rightarrow I \mid (U, V) \in M\}$$

of continuous functions. It remains to prove that $\mathcal{F}$ can distinguish points from closed sets of X .

For this purpose, let A be any closed set of X and p be an arbitrary point in the point set $X \setminus A$. Since $\mathcal{B}$ is a basis, there exists a $V \in \mathcal{B}$ such that $p \in V \subset X \setminus A$. Since X is regular, p has a closed neighborhood N contained in V . Again, since $\mathcal{B}$ is a basis there exists a

$U \in \mathcal{B}$ such that $p \in U \subset N$. Since N is closed and $N \subset V$, we have $\text{Cl}(U) \subset V$. Hence $(U, V) \in M$. Then, it is clear that

$$f_{(U, V)}(p) = 0, \quad f_{(U, V)}(A) = 1.$$

Hence $\mathcal{F}$ can distinguish points from closed sets of X . $\parallel$

According to Exercise 1B and (II, Ex. 5J), I^N is a regular Fréchet space satisfying the second axiom of countability. Hence, we have the following theorem.

THEOREM 6.7. *A space X is a regular Fréchet space satisfying the second axiom of countability iff X is homeomorphic to a subspace of the Hilbert cube I^N .*

EXERCISES

- 6A. Let X denote the set of all real numbers with the half-open interval topology, i.e. the smallest topology containing all half-open intervals $[a, b)$. Prove:
- X is regular and normal.
 - $X \times X$ is not normal.
- 6B. A set E in a space X is called a G_δ iff it is the intersection of the members of a countable family of open sets. Prove that if f is a continuous real function on a space X , then the inverse image $f^{-1}[a, b]$ of a closed interval $[a, b]$ is a G_δ in X .
- 6C. A set E in a space X is called an F_σ iff it is the union of the members of a countable family of closed sets. Prove that E is an F_σ iff $X \setminus E$ is a G_δ .

7. EXTENSION THEOREMS

Let X and Y be given spaces and let E be a given subspace of X . We recall the extension problem of any given map

$$g : E \rightarrow Y$$

over the space X formulated in (II, § 4) as follows: to determine whether or not g has an extension over X and to find an extension of g over X if such exists.

A few special cases of this important problem have been solved in the previous sections. As examples, we have the following results:

- (1) Let X be a completely regular space and E be a subspace of X

which consists of a closed set F of X and a point $p \in X \setminus F$. If $g : E \rightarrow I$ denotes the map defined by

$$g(x) = \begin{cases} 0, & (\text{if } x = p), \\ 1, & (\text{if } x \in F), \end{cases}$$

then g has an extension $f : X \rightarrow I$.

(2) Let X be a normal space and E be the union of two disjoint closed subspaces A and B of X . If $g : E \rightarrow I$ denotes the map defined by

$$g(x) = \begin{cases} 0, & (\text{if } x \in A), \\ 1, & (\text{if } x \in B), \end{cases}$$

then g has an extension $f : X \rightarrow I$ according to Urysohn's lemma.

(3) Let X be a connected space and $E = Y$ be a disconnected subspace of X . Then the identity map $g : E \rightarrow Y$ has no extension over X by (4.5).

(4) Let X be a compact space and $E = Y$ be a non-compact subspace of X . Then the identity map $g : E \rightarrow Y$ has no extension over X by (2.3).

(5) Let X be a Hausdorff space and $E = Y$ be a subspace of X which is not closed. Then the identity map $g : E \rightarrow Y$ has no extension over X according to (1.4).

In the present section, we will study two cases where the extension problem is trivial, i.e. the answer is always affirmative.

THEOREM 7.1. *The subspace E is a retract of the space X iff every map $g : E \rightarrow Y$ from E into any space Y has an extension over X .*

Proof: Necessity. Assume that E is a retract of X . Then there exists a retraction $r : X \supset E$. Obviously, the composition $f = g \circ r$ is an extension of g over X .

Sufficiency. Assume that the condition holds. Take $Y = E$ and consider the identity map $g : E \rightarrow Y$. According to the condition, g has an extension $r : X \rightarrow Y = E$ which is evidently a retraction of X onto E . ||

THEOREM 7.2 (TIETZ'S EXTENSION THEOREM). *If E is a closed subspace of a normal space X , then every map $g : E \rightarrow I$ has an extension over X .*

Proof. For any two disjoint closed sets A and B in X , let us denote by $\chi_{A,B}$ a characteristic function whose existence follows from the normality of X according to Urysohn's lemma. $\chi_{A,B}$ is a map of X into I such that

$$\chi_{A,B}(x) = \begin{cases} 0, & (\text{if } x \in A), \\ 1, & (\text{if } x \in B). \end{cases}$$

Now let $g : E \rightarrow I$ be an arbitrarily given map. For each $n = 0, 1, 2, \dots$, we shall define two maps

$$g_n : E \rightarrow I, \quad f_n : X \rightarrow I$$

as follows. Let $g_0 = g$. Suppose that g_n has already been defined. Let

$$A_n = \{x \in E \mid g_n(x) \leq (1/3)(2/3)^n\},$$

$$B_n = \{x \in E \mid g_n(x) \geq (2/3)(2/3)^n\}.$$

Since E is closed in X , A_n and B_n are two disjoint closed sets of the normal space X . Pick a characteristic function χ_{A_n, B_n} and let

$$f_n = \frac{1}{3} \left(\frac{2}{3}\right)^n \chi_{A_n, B_n} : X \rightarrow I.$$

Next, define $g_{n+1} : E \rightarrow I$ by taking

$$g_{n+1}(x) = g_n(x) - f_n(x)$$

for every $x \in E$. To justify this definition, one must verify that

$$f_n(x) \leq g_n(x)$$

for every $x \in E$. For this purpose, let $x \in E$. If $x \in A_n$, then $f_n(x) = 0$ and hence $f_n(x) \leq g_n(x)$. If $x \notin A_n$, then

$$g_n(x) > (1/3)(2/3)^n.$$

Since $f_n(x) \leq (1/3)(2/3)^n$, we have $f_n(x) \leq g_n(x)$. This completes the inductive definition of the maps g_n and f_n .

Since, for each $x \in X$, we have

$$0 \leq f_n(x) \leq \frac{1}{3} \left(\frac{2}{3}\right)^n,$$

it follows that the series $\sum f_n(x)$ is uniformly convergent on X . Since f_n is a continuous function for each n , it follows from classical analysis that the limit

$$\lim_{n \rightarrow \infty} [f_0(x) + f_1(x) + \dots + f_n(x)]$$

exists for every $x \in X$ and defines a continuous function

$$f : X \rightarrow I.$$

It remains to prove that $f|E = g$. For this purpose, let x be an arbitrary point of E and we will prove $f(x) = g(x)$. In order to do this, let us first prove by induction that

$$0 \leq g_n(x) \leq \left(\frac{2}{3}\right)^n.$$

First, the inequalities hold obviously for $n = 0$. Now assume the inequalities for n . By definition, we have

$$0 \leq g_{n+1}(x) = g_n(x) - f_n(x).$$

If $x \in B_n$, then $f_n(x) = (1/3)(2/3)^n$ and hence

$$g_{n+1}(x) \leq \left(\frac{2}{3}\right)^n - \frac{1}{3} \left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^{n+1}.$$

If $x \notin B_n$, then $g_n(x) < (2/3)(2/3)^n$ and hence

$$g_{n+1}(x) = g_n(x) - f_n(x) \leq g_n(x) < \left(\frac{2}{3}\right)^{n+1}.$$

This completes the inductive proof of the inequalities.

Finally, since $g_{r+1}(x) = g_r(x) - f_r(x)$ by definition, it follows that

$$f_r(x) = g_r(x) - g_{r+1}(x).$$

Summing for $r = 0, 1, \dots, n$, we obtain

$$f_0(x) + f_1(x) + \dots + f_n(x) = g(x) - g_{n+1}(x).$$

Now, let $n \rightarrow \infty$. The left member gives $f(x)$ while the right member approaches $g(x)$. Hence $f(x) = g(x)$. $\square$

EXERCISES

- 7A. A closed subspace E of a space X is said to have the *extension property* in X with respect to a space Y iff every map $g: E \rightarrow Y$ has an extension over X . A space Y is said to be *solid* iff every closed subspace E of a normal space X has the extension property in X with respect to Y . By Tietze's extension theorem, every closed interval of real numbers is a solid space under the usual topology. Prove:
- (i) The topological product of any family of solid spaces is solid.
Hence, the cubes are solid.
 - (ii) Any retract of a solid space is solid.
- 7B. Let E denote the subspace of the unit interval $I = [0, 1]$ which consists of the two end points 0 and 1. Prove that a space X is pathwise connected iff E has the extension property in I with respect to X .

8. COMPACTIFICATION

In studying a non-compact space X , it is often helpful to construct a compact space Y which contains X as dense subspace. The device is called compactification. Precisely, by a *compactification* of a space X , we mean an imbedding

$$h: X \rightarrow Y$$

of the space X into a compact space Y such that $h(X)$ is dense in Y .

The simplest compactification of a non-compact space X is the *one-point compactification*

$$h: X \rightarrow X^*$$

obtained by adjoining a single point ∞ to the space X as follows. The points of X^* are those of X together with the point ∞ which may be any element not in X . The topology of X^* consists of the open sets of X and all subsets U of X^* such that $X^* \setminus U$ is a closed compact subset of X . One can easily verify that the axioms for open sets are satisfied.

THEOREM 8.1. *If X is non-compact, then the inclusion function $h: X \subset X^*$ is a compactification of X .*

Proof. By the definition of the topology of X^* , it is clear that the inclusion function $h: X \subset X^*$ is an imbedding. It remains to prove that X^* is compact and that X is dense in X^* .

Let α be an arbitrary open cover of X^* . Then there is a member U_0 of α which contains the point ∞ . This implies that $X^* \setminus U_0$ is a compact subset of X . Hence there exists a finite number of members of α , say $U_1, \dots, U_n$, such that

$$X^* \setminus U_0 \subset (U_1 \cup \dots \cup U_n).$$

This implies that $\{U_0, U_1, \dots, U_n\}$ is a finite subcover of α . Therefore, X^* is compact.

To prove that X is dense in X^* , we have to use the hypothesis that X is non-compact. For this purpose, let U denote an arbitrary non-empty open set of X^* . If the point ∞ is not in U , then $U \subset X$. If the point ∞ is in U , then $X^* \setminus U$ is a compact subset of X . Since X is non-compact, this implies that U contains at least one point of X . Hence, X is dense in X^* . ||

Although the one-point compactification of a non-compact space X is simple and elegant, it is unsatisfactory unless the given space X is locally compact. In fact, we have the following proposition.

PROPOSITION 8.2. *The space X^* is a Hausdorff space iff X is a locally compact Hausdorff space.*

Proof: Necessity. If X^* is a Hausdorff space, then the open subspace $X = X^* \setminus \{\infty\}$ of the compact Hausdorff space X^* is a locally compact Hausdorff space.

Sufficiency. Assume that X is a locally compact Hausdorff space. We will prove that X^* is a Hausdorff space. For this purpose, let a and b be any two distinct points of X^* . If none of the points a and b is ∞ , then there are open sets U and V of X such that

$$a \in U, \quad b \in V, \quad U \cap V = \emptyset.$$

By the definition of the topology of X^* , U and V are also open sets of X^* . On the other hand, if one of the points a and b is the point ∞ , we may assume without loss of generality that $a = \infty$ and $b \in X$. Since X is locally compact, b has a compact neighborhood K in X . Let

$$U = X^* \setminus K, \quad V = \text{Int}(K).$$

As a compact set in a Hausdorff space X , K is closed. Hence U and V are open sets of X^* . Obviously, we have

$$a \in U, \quad b \in V, \quad U \cap V = \emptyset.$$

Hence, X^* is a Hausdorff space. $\square$

In the remainder of this section, we are concerned with another important compactification, namely, the *Stone-Čech compactification* of a Tychonoff space.

Let X be any given Tychonoff space and consider the family M of all continuous real functions $f_\mu: X \rightarrow I$ from X into the closed unit interval I . According to the Tychonoff imbedding theorem (6.4), the Cartesian product

$$f: X \rightarrow I^M$$

of the family $M = \{f_\mu\}$ is an imbedding. By (2.11), I^M is compact. Let $\beta(X)$ denote the closure of the image $f(X)$ in I^M and let

$$h: X \rightarrow \beta(X)$$

denote the map defined by f . Then it is clear that $h: X \rightarrow \beta(X)$ is a compactification of X called the *Stone-Čech compactification* of X .

The Stone-Čech compactification has an important property given by the following theorem.

THEOREM 8.3. *If X is a Tychonoff space, then every map $\phi: X \rightarrow Y$ of X into a compact Hausdorff space Y has an extension over $\beta(X)$. Precisely, there exists a map $\psi: \beta(X) \rightarrow Y$ such that $\phi = \psi \circ h$ holds in the following triangle*

$$\begin{array}{ccc} X & \xrightarrow{h} & \beta(X) \\ & \searrow \phi & \swarrow \psi \\ & Y & \end{array}$$

Proof. Consider the families of continuous real functions

$$M = \text{Map}(X, I), \quad N = \text{Map}(Y, I)$$

Then the map $\phi: X \rightarrow Y$ induces a function

$$\phi^\#: N \rightarrow M$$

defined by $\phi^\#(\eta) = \eta \circ \phi$ for every $\eta: Y \rightarrow I$ in N .

Next, consider the topological products

$$f: X \rightarrow I^M, \quad g: Y \rightarrow I^N$$

of the families M and N . Since both X and Y are Tychonoff spaces, f and g are imbeddings and define the Stone-Čech compactifications

$$h: X \rightarrow \beta(X), \quad k: Y \rightarrow \beta(Y)$$

respectively. Since Y is compact, it follows that k is a homeomorphism.

Define a function

$$\phi_*: I^M \rightarrow I^N$$

by taking $\phi_*(\alpha) = \alpha \circ \phi^\#: N \rightarrow I$ for every $\alpha: M \rightarrow I$ in I^M . For each $\eta: Y \rightarrow I$ in N , let

$$p_\eta: I^N \rightarrow I$$

denote the natural projection of the product space I^N onto the η -th coordinate space. Then, the composed function

$$p_\eta \circ \phi_*: I^M \rightarrow I$$

gives $(p_\eta \circ \phi_*)(\alpha) = p_\eta(\alpha \circ \phi^\#) = \alpha[\phi^\#(\eta)] = \alpha(\eta \circ \phi)$ for each $\alpha \in I^M$. Hence, $p_\eta \circ \phi_*$ is the natural projection of I^M onto the $(\eta \circ \phi)$ -th coordinate space. This implies that ϕ_* is continuous.

Consider the following rectangle of maps:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \downarrow f & & \downarrow g \\ I^M & \xrightarrow{\phi_*} & I^N \end{array}$$

This rectangle is commutative, that is to say, $\phi_* \circ f = g \circ \phi$. To verify this, let x be an arbitrary point of X . We have to prove that the two points $\phi_*[f(x)]$ and $g[\phi(x)]$ of I^N are coincident. For this purpose, let $\eta: Y \rightarrow I$ be an arbitrary element in N . Then, the η -th coordinates of the points $\phi_*[f(x)]$ and $g[\phi(x)]$ are as follows:

$$p_\eta\{\phi_*[f(x)]\} = (p_\eta \circ \phi_*)[f(x)] = (\eta \circ \phi)(x),$$

$$p_\eta\{g[\phi(x)]\} = \eta[\phi(x)] = (\eta \circ \phi)(x).$$

Hence $\phi_*[f(x)] = g[\phi(x)]$. This proves the commutativity of the rectangle.

As a consequence of the commutativity $\phi_* \circ f = g \circ \phi$, the map ϕ_* send $f(X)$ into $g(X)$. By the continuity of ϕ_* , this implies that ϕ_* carries $\beta(X)$ into $\beta(Y)$ and hence defines a map

$$\beta(\phi): \beta(X) \rightarrow \beta(Y).$$

Therefore, we obtain a commutative rectangle:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \downarrow h & & \downarrow k \\ \beta(X) & \xrightarrow{\beta(\phi)} & \beta(Y) \end{array}$$

Since k is a homeomorphism, we may define a map

$$\psi: \beta(X) \rightarrow Y$$

by taking $\psi = k^{-1} \circ \beta(\phi)$. The commutativity of the rectangle gives $\phi = \psi \circ h$. ||

By a *Hausdorff compactification* of a space X , we mean a compactification $f: X \rightarrow Y$ where Y is a Hausdorff space.

As an immediate consequence of (8.3), we have the following corollary.

COROLLARY 8.4. If $j: X \rightarrow Y$ is any Hausdorff compactification of a Tychonoff space X , then there exists a map $\psi: \beta(X) \rightarrow Y$ such that $j = \psi \circ h$, where $h: X \rightarrow \beta(X)$ denotes the Stone-Čech compactification of X .

Because of (8.4), the Stone-Čech compactification of X is sometimes referred to as the *maximal (Hausdorff) compactification* of X .

EXERCISES

- 8A. If X is a compact space, prove that the point ∞ is an *isolated point* of the one-point compactification X^* of X , that is, $\{\infty\}$ is both open and closed in X^* .
- 8B. Construct a compactification of the space R of real numbers by adjoining two new points $+\infty$ and $-\infty$. The compact space obtained is called the *extended real line*.
- 8C. Prove that the one-point compactification of the Euclidean n -space R^n is homeomorphic to the n -sphere S^n .

9. HEREDITARY PROPERTIES

In (II, § 3) we introduced topological equivalence and topological properties of spaces. In (II, § 7) we introduced the notions of homotopy properties and isotopy properties. We also observed that every homotopy property is an isotopy property and that every isotopy property is a topological property. As examples, we proved that contractability and connectedness are homotopy properties of spaces.

To investigate the problem of whether or not the other properties studied in this chapter are homotopy properties or isotopy properties, let us first introduce the notion of hereditary properties and weakly hereditary properties.

A property P of spaces is said to be *hereditary* iff every subspace of a space with P also has P ; it is said to be *weakly hereditary* iff every closed subspace of a space with P also has P . As examples, the following topological properties of a space X are weakly hereditary:

- (A) X is a Fréchet space.
- (B) X is a Hausdorff space.
- (C) X is a regular space.
- (D) X is a completely regular space.
- (E) X is a discrete space.

- (F) X is an indiscrete space.
- (G) The first axiom of countability is satisfied in X .
- (H) The second axiom of countability is satisfied in X .
- (I) X can be imbedded in a given space Y .
- (J) X is a normal space.
- (K) X is a compact space.
- (L) X is a paracompact space.
- (M) X is a locally compact space.

The first nine properties (A)–(I) listed above are also hereditary.

THEOREM 9.1. *If P is a weakly hereditary topological property such that every singleton space has P and that there exists a space X which does not have P , then P is not a homotopy property.*

Proof. Let X denote a space which does not have P . Consider the cone $\text{Con}(X)$ over X which is the quotient space obtained by identifying the top $X \times 1$ of the cylinder $X \times I$ to a single point v , called the vertex of $\text{Con}(X)$. Since X is homeomorphic to the bottom $X \times 0$ of $\text{Con}(X)$ which is a closed subspace of $\text{Con}(X)$ and since P is a weakly hereditary topological property which X does not have, it follows that $\text{Con}(X)$ cannot have P . Since $\text{Con}(X)$ is contractible by (II, Ex. 7B), the inclusion map

$$i: v \rightarrow \text{Con}(X)$$

is a homotopy equivalence by (II, 7.6 and 7.7). Since the singleton space v has P while $\text{Con}(X)$ does not have P , P is not a homotopy property. $\parallel$

Since every singleton space X has all of the properties (A)–(M) and since none of these properties prevails in all spaces, we have the following corollary of (9.1).

COROLLARY 9.2. *None of the properties (A)–(M) is a homotopy property.*

The next question is whether or not these properties (A)–(M) are isotopy properties. For this purpose, we will establish the following theorem.

THEOREM 9.3. *Every hereditary topological property of spaces is an isotopy property.*

Proof. Let P denote any hereditary topological property of spaces. Assume that $f: X \rightarrow Y$ is an isotopy equivalence and that the space X has the property P . It suffices to prove that Y also has P .

By the definition of an isotopy equivalence, there exists an imbedding

$g: Y \rightarrow X$ such that the composed imbeddings $g \circ f$ and $f \circ g$ are isotopic to the identity maps on X and Y respectively. The image $g(Y)$ is a subspace of X . Since P is hereditary, this implies that $g(Y)$ has P . As an imbedding, g is an homeomorphism of Y onto $g(Y)$. Since P is a topological property and $g(Y)$ has P , it follows that Y also has P . ||

Since the first nine properties (A)–(I) are hereditary, we have the following corollary of (9.3).

COROLLARY 9.4. *Each of the properties (A)–(I) is an isotopy property.*

It remains to see if the properties (J)–(M) are isotopy properties.

PROPOSITION 9.5. *Compactness is not an isotopy property.*

Proof. By (II, 7.8), the closed unit interval I is isotopically equivalent to the open unit interval J . Since I is compact while J is non-compact, compactness cannot be an isotopy property. ||

PROPOSITION 9.6. *Local compactness is not an isotopy property.*

Proof. By (II, Ex. 7E), the topological powers I^M and R^M are isotopically equivalent for every set M . If M is an infinite set, R^M is not locally compact by (Ex. 2E) while I^M is compact and hence locally compact. Therefore, local compactness cannot be an isotopy property. ||

For the remaining question of whether the properties (J) and (L) are isotopy properties, see the exercises 9D and 9E.

EXERCISES

- 9A. A space X is said to be a *Lindelöf space* iff every open cover of X has a countable subcover. Prove that being a Lindelöf space is a weakly hereditary topological property and hence not a homotopy property.
- 9B. The *inductive dimension* $\dim X$ of a space X is defined as follows: $\dim X = -1$ iff X is empty, and $\dim X \leq n$ iff for every point $p \in X$ and every neighborhood U of p there exists an open neighborhood $V \subset U$ of p such that $\dim(\partial V) \leq n - 1$, where ∂V denotes the boundary $\bar{V} \setminus V$ of V in X , and finally

$$\dim X = \inf \{n \mid \dim X \leq n\}.$$

Prove that, for a given integer $n \geq 0$, $\dim X \leq n$ is a hereditary topological property. Then, deduce that $\dim X$ is an isotopy property but not a homotopy property. [Hu 1].

9C. A topological property of spaces is said to be an *open property* iff it is inherited by open subspaces and preserved by open maps. Verify that the following topological properties are open properties:

- (a) separability,
- (b) local connectedness,
- (c) local pathwise connectedness.

Prove that nontrivial open properties are not homotopy properties. [Gottlieb].

9D. Prove that normality is not an isotopy property, by the following method or otherwise. Consider the set Ω' of all ordinal numbers not greater than the first uncountable ordinal Ω . Give Ω' the order topology which is the smallest topology in Ω' containing all sets of the form $\{x \in \Omega' \mid x < a\}$ or $\{x \in \Omega' \mid a < x\}$ for some $a \in \Omega'$. Let

$$X = \Omega' \times I, \quad Y = X \setminus \{(\Omega, 1)\}.$$

Then it remains to prove the following three assertions: [Gottlieb 1].

- (a) $X \cong Y$,
- (b) X is normal,
- (c) Y is not normal.

9E. By using the spaces in the preceding exercise or otherwise, prove that a topological property P is not an isotopy property if it satisfies the following two conditions:

- (a) Every compact Hausdorff space has P .
- (b) Every regular space possessing P is normal.

In particular, paracompactness is not an isotopy property. Also, being a Lindelöf space is not an isotopy property. [Gottlieb 1].

Chapter IV: METRIC SPACES

The concept of topological spaces is motivated by a special kind of these spaces called metric spaces with topology defined by given distance functions. The present chapter is devoted to the properties of metric spaces and the metrizable of topological spaces. Because most of the spaces which occur naturally in modern analysis are pseudo-metric rather than metric, and because, as we shall see in this chapter, pseudo-metric spaces have the same properties as metric spaces except for being Fréchet spaces and the consequences of this, we will work with pseudo-metrics as long as possible. Besides, pseudo-metrics were found very convenient in solving the metrization problem.

1. DISTANCE FUNCTIONS

By a *pseudo-metric* in a set X , we mean a real-valued function

$$d: X \times X \rightarrow R$$

satisfying the following conditions:

(M1) *The triangle inequality*: For any three points a, b, c in X , we have

$$d(a, c) + d(b, c) \geq d(a, b).$$

(M2) For every point x in X , we have

$$d(x, x) = 0.$$

LEMMA 1.1. *If $d: X^2 \rightarrow R$ is any pseudo-metric in a set X , then we have*

(M3) $d(a, b) \geq 0$

(M4) $d(a, b) = d(b, a)$

for any two points a and b in X .

Proof. To establish (M3), we apply (M1) to the points a, a, b of X and get

$$2d(a, b) = d(a, b) + d(a, b) \geq d(a, a).$$

By (M2), we have $d(a, a) = 0$. Hence, we obtain $d(a, b) \geq 0$.

To establish (M4), we apply (M1) to the points a, b, a of X and obtain

$$d(a, a) + d(b, a) \geq d(a, b).$$

Because of $d(a, a) = 0$, this implies $d(b, a) \geq d(a, b)$. Similarly, we can prove $d(a, b) \geq d(b, a)$. Hence we obtain (M4). $\parallel$

By a *metric* in a set X , we mean a pseudo-metric $d: X^2 \rightarrow R$ in X which satisfies the following condition:

(M5) For any two points a and b in X , $d(a, b) = 0$ implies $a = b$.

EXAMPLES OF METRICS AND PSEUDO-METRICS:

(1) *The usual metric in R .* For the real line R , define a function $d: R^2 \rightarrow R$ by taking

$$d(a, b) = |b - a|$$

for any two real numbers a and b , where $|x|$ denotes the absolute value of the real number x . The conditions (M1), (M2), and (M5) can be easily verified by the properties of the absolute values of real numbers. Hence d is a metric in the real line R called the *usual metric* in R .

(2) *The discrete metric.* Let X denote an arbitrary set. Define a function $d: X^2 \rightarrow R$ by taking

$$d(a, b) = \begin{cases} 0, & (\text{if } a = b), \\ 1, & (\text{if } a \neq b), \end{cases}$$

for any two points a and b in X . The conditions (M1), (M2), and (M5) are obviously satisfied. Hence d is a metric in the set X called the *discrete metric* in X .

(3) *The trivial pseudo-metric.* Let X denote an arbitrary set. Define a function $d: X^2 \rightarrow R$ by taking

$$d(a, b) = 0$$

for any two points a and b in X . The conditions (M1) and (M2) are obviously satisfied. Hence d is a pseudo-metric in the set X called the *trivial pseudo-metric* in X . Unless X is a singleton, the condition (M5) fails to be true and hence d is not a metric in X .

Now let us consider an arbitrary pseudo-metric

$$d: X^2 \rightarrow R$$

in a given set X . For any point p of X and any positive real number r , the subset

$$N_r(p) = \{x \in X \mid d(p, x) < r\}$$

is called the r -neighborhood of the point p in X with respect to the pseudo-metric d .

Define a collection τ of subsets of X as follows. For an arbitrary subset U of X , U is in τ iff, for any point $p \in U$, there exists a positive real number r such that the inclusion

$$N_r(p) \subset U$$

is true. We will establish the following proposition.

PROPOSITION 1.1. *The collection τ of subsets of X is a topology in X .*

Proof. We have to verify the four axioms (OS1)–(OS4) in (II, § 1). Since the first three axioms are obviously true, it remains to verify (OS4). For this purpose, let V and W denote any two sets in the collection τ . We will prove that the intersection

$$U = V \cap W$$

is also in the collection τ . Let p denote any point in U . Since V and W are in τ , it follows from the definition of τ that there are positive real numbers s and t with $N_s(p) \subset V$ and $N_t(p) \subset W$. Let

$$r = \min\{s, t\}$$

denote the smaller of the two positive real numbers s and t . Then we have $r > 0$ and

$$N_r(p) \subset N_s(p) \cap N_t(p) \subset V \cap W = U.$$

Hence U is in τ by definition. $\parallel$

This topology τ in X will be referred to as the *topology defined by the pseudo-metric d* . As usual, the sets in τ are called *open sets*.

PROPOSITION 1.2. *The collection*

$$\beta = \{N_r(p) \mid p \in X \text{ and } r > 0\}$$

of r -neighborhoods $N_r(p)$ for all $p \in X$ and all positive real numbers r is a basis of the topology τ .

Proof. First let us prove that $N_r(p)$ is open for every $p \in X$ and every $r > 0$. For this purpose, let q denote any point of $N_r(p)$. Then we have

$$s = d(p, q) < r.$$

Thus we obtain a positive real number

$$t = r - s > 0.$$

To prove $N_t(q) \subset N_r(p)$, let x denote any point of $N_t(q)$. Then we have

$$d(p, x) \leq d(p, q) + d(q, x) = s + d(q, x) < s + t = r$$

because of (M1) and (M4). This implies $d(p, x) < r$ and hence $x \in N_r(p)$. Thus we have proved $\beta \subset \tau$.

Next let U denote any open set in τ and let p stand for any point in U . According to the definition of τ , there exists a positive real number r such that the set

$$B_p = N_r(p)$$

of β is contained in U . On the other hand, it follows from (M2) that the point p is contained in B_p . Hence, we obtain

$$U = \bigcup_{p \in U} B_p$$

for every $U \in \tau$. This proves that β is a basis of τ . ||

Because of (1.2), the sets $N_r(p)$ in β are expected to play important rôles in the topology defined by a pseudo-metric d . We will also call $N_r(p)$ the *open ball* with *center* p and *radius* r .

Now let us consider any two pseudo-metrics

$$d, d': X^2 \rightarrow R$$

in the same set X . Then we say that d and d' are *equivalent*, in symbol

$$d \sim d',$$

iff both define the same topology in the set X .

For every point $p \in X$ and every real number $r > 0$, let $N_r(p)$ and $N'_r(p)$ denote the r -neighborhoods of p with respect to the pseudo-metrics d and d' respectively. We will establish the following proposition.

PROPOSITION 1.3. *Two pseudo-metrics d and d' in a set X are equivalent iff, for any point $p \in X$ and any real number $r > 0$, there exist positive real numbers s and t satisfying*

$$N_s(p) \subset N'_r(p), \quad N'_t(p) \subset N_r(p).$$

Proof: Necessity. Assume $d \sim d'$. Then the set $N'_r(p)$ is an open set in the topology τ defined by d . Since p is in $N'_r(p)$ by the condition (M2) for d' , there exists a real number $s > 0$ satisfying $N_s(p) \subset N'_r(p)$. Similarly, one can establish the existence of a real number $t > 0$ satisfying $N'_t(p) \subset N_r(p)$.

Sufficiency. Assume that the condition is satisfied. We will prove that the topology τ defined by d coincides with the topology τ' defined by d' . For this purpose, let U denote any open set in the topology τ and let p stand for any point in U . According to the definition of τ , there exists a real number $r > 0$ such that $N_r(p) \subset U$ holds. Because of our condition, there exists a real number $t > 0$ satisfying

$$N'_t(p) \subset N_r(p) \subset U.$$

According to the definition of τ' , this implies $U \in \tau'$. Thus we have proved $\tau \subset \tau'$. Similarly, one can prove $\tau' \subset \tau$. Hence we obtain $\tau = \tau'$ and $d \sim d'$. $\parallel$

As an easy consequence of (1.3), we have the following corollary.

COROLLARY 1.4. *Let d and d' be any two equivalent pseudo-metrics in a given set X' . If d is a metric, then so is d' .*

Proof. Let a and b denote any two distinct points of X . It suffices to prove $d'(a, b) > 0$. Since d is a metric, it follows from the conditions (M3) and (M5) for d that we have

$$r = d(a, b) > 0.$$

Since $d \sim d'$, there exists a positive real number t satisfying

$$N'_t(a) \subset N_r(a)$$

because of (1.3). Since b is not in $N_r(a)$, it can never be in $N'_t(a)$. This implies

$$d'(a, b) \geq t > 0.$$

Hence d' is a metric. $\parallel$

A pseudo-metric $d: X^2 \rightarrow R$ is said to be *bounded* iff there exists a real number k such that $d(a, b) \leq k$ holds for every pair (a, b) of points in X .

PROPOSITION 1.5. *Every pseudo-metric in any given set is equivalent to a bounded pseudo-metric.*

Proof. Let $d: X^2 \rightarrow R$ denote an arbitrary pseudo-metric in any given set X . Define a function $d': X^2 \rightarrow R$ by taking

$$d'(a, b) = \min\{1, d(a, b)\}$$

for any two points a and b in X .

To prove that d' is a pseudo-metric, we have to verify (M1) and (M2). To verify (M1), consider arbitrary points a, b, c of X . In case $d(a, c) \leq 1$ and $d(b, c) \leq 1$, we have

$$d'(a, b) \leq d(a, b) \leq d(a, c) + d(b, c) = d'(a, c) + d'(b, c);$$

otherwise, we have

$$d'(a, b) \leq 1 \leq d'(a, c) + d'(b, c).$$

Hence (M1) holds in both cases. To verify (M2), let x denote an arbitrary point in X . Then we have

$$d'(a, a) = \min\{1, d(a, a)\} = \min\{1, 0\} = 0.$$

Hence (M2) is also true. This proves that d' is a pseudo-metric in X .

To prove $d \sim d'$, consider any point $p \in X$ and an arbitrary real number $r > 0$. Choose any two positive real numbers s and t less than $\min\{1, r\}$. Then it follows immediately from the definition of d' that we have the following inclusions

$$N_s(p) \subset N_r'(p), \quad N_t'(p) \subset N_r(p).$$

According to (1.3), d and d' are equivalent.

Finally, since $d'(a, b) \leq 1$ holds for every pair (a, b) of points in X , d' is bounded. This completes the proof of (1.5). $\parallel$

EXERCISES

1A. For an arbitrary pseudo-metric d in any given set X , prove

$$|d(a, c) - d(b, c)| \leq d(a, b)$$

$$d(x_1, x_n) \leq \sum_{i=1}^{n-1} d(x_i, x_{i+1})$$

for arbitrary points $a, b, c, x_1, x_2, \dots, x_n$ in the set X .

1B. For any subset A of a given set X , prove that the restriction

$$e = d \upharpoonright A^2: A^2 \rightarrow R$$

of every pseudo-metric $d: X^2 \rightarrow R$ is a pseudo-metric in A , called the *restriction of d in A* . If d is a metric, then so is e .

1C. For the real line R , define a function

$$\tilde{d}: R^2 \rightarrow R$$

as follows. Let a and b denote any two real numbers. If one and only one of the real numbers a, b is positive, define

$$d(a, b) = 1 + |b - a|;$$

otherwise, define

$$d(a, b) = |b - a|.$$

Prove that d is a metric in R which is not equivalent to the usual metric.

For the open unit interval $J = (0, 1)$ of real numbers, define a function

$$d: J^2 \rightarrow R$$

by taking

$$d(a, b) = |b^{-1} - a^{-1}|$$

for every two real numbers a and b in J . Prove that d is a metric equivalent to the usual metric in R .

For the plane $P = R \times R$ of analytic geometry, prove that the functions

$$d, e, f: P^2 \rightarrow R$$

defined by

$$d[(x_1, y_1), (x_2, y_2)] = \sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2]}$$

$$e[(x_1, y_1), (x_2, y_2)] = |x_1 - x_2| + |y_1 - y_2|$$

$$f[(x_1, y_1), (x_2, y_2)] = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

for any two points (x_1, y_1) and (x_2, y_2) of the plane P are equivalent metrics in P . Compare the balls in P with respect to these metrics.

A map $f: M \rightarrow M$ of the subspace

$$M = \{w \in R \mid w \geq 0\}$$

of the real line R is said to be *sub-additive* iff

$$f(u + v) \leq f(u) + f(v)$$

holds for all real numbers u and v in M . Verify that the function $f: M \rightarrow M$ defined by

$$f(w) = \frac{w}{1 + w}$$

is sub-additive. For every pseudo-metric d in any given set X , prove that the function $e: X^2 \rightarrow R$ defined by

$$e(a, b) = f[d(a, b)]$$

for any two points a and b of X is a pseudo-metric in X equivalent to d if f is a sub-additive function.

For an arbitrary pseudo-metric d in a given set X , prove that the set

$$\{x \in X \mid d(p, x) > r\}$$

for any given point $p \in X$ and any given real number $r > 0$ is in the topology τ defined by d . Hence,

$$\{x \in X \mid d(p, x) \leq r\}$$

is a closed set, called the *closed ball* with *center* p and *radius* r .

2. METRIC SPACES

By a *pseudo-metric space*, we mean a set X furnished with a pseudo-metric

$$d: X^2 \rightarrow R$$

and the topology τ defined by d . In other words, a pseudo-metric space is a topological space X together with a pseudo-metric d in X which defines its topology τ .

A pseudo-metric space X will be called a *metric space* iff its pseudo-metric d is a metric.

PROPOSITION 2.1. *A pseudo-metric space is a Fréchet space iff it is a metric space.*

Proof: Necessity. Assume that a given pseudo-metric space X is a Fréchet space. We will prove that the pseudo-metric d in X must be a metric. For this purpose, consider any two distinct points a and b in X . Since X is a Fréchet space, the singleton subset $\{b\}$ of X is closed and hence its complement

$$U = X \setminus \{b\}$$

is open. Since a and b are distinct, we have $a \in U$. Since the topology τ of X is defined by the pseudo-metric d , it follows that there exists a positive real number r satisfying $N_r(a) \subset U$. Since b is not in U , this implies $b \notin N_r(a)$. Hence we obtain

$$d(a, b) \geq r > 0.$$

This proves that d is a metric.

Sufficiency. Assume that X is a metric space with metric d . To prove that X is a Fréchet space, let a denote an arbitrary point in X . It suffices to prove that the subset

$$V = X \setminus \{a\}$$

of X is open. For this purpose, let b denote any point in V . Since d is a metric, we have

$$r = d(a, b) > 0.$$

Then it follows that the r -neighborhood $N_r(b)$ is contained in V . According to the definition of the topology defined by d , this implies that V is open. $\parallel$

EXAMPLES OF METRIC AND PSEUDO-METRIC SPACES:

(1) *The real line R .* The set R of all real numbers together with the usual metric d in R constitutes a metric space. One can easily verify that the topology τ in R defined by the usual metric d is the usual topology in R . This metric space R will still be called the *real line*.

(2) *The discrete metric space X .* An arbitrarily given set X together with the discrete metric d in X constitutes a metric space. For any point $p \in X$ and any positive real number $r \leq 1$, we have $N_r(p) = \{p\}$. Hence $\{p\}$ is an open set in the topology τ defined by d . This proves that X is a discrete space.

(3) *The indiscrete pseudo-metric space X .* An arbitrarily given set X together with the trivial pseudo-metric in X constitutes a pseudo-metric space. For any point $p \in X$ and any positive real number r , we have $N_r(p) = X$. Hence X is the only non-empty open set in the topology τ defined by d . This proves that X is an indiscrete space.

(4) *The Euclidean n -space R^n .* For the Cartesian power $X = R^n$ of the real line R , define a function $d: X^2 \rightarrow R$ by taking

$$d(a, b) = \sqrt{\sum_{i=1}^n (a_i - b_i)^2}$$

for any two points $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ in $X = R^n$. According to the Schwarz inequality, we have

$$d(a, b) d(b, c) \geq \sum_{i=1}^n (a_i - b_i)(b_i - c_i)$$

and therefore

$$\begin{aligned} [d(a, b) + d(b, c)]^2 &\geq \sum_{i=1}^n (a_i - b_i)^2 + 2 \sum_{i=1}^n (a_i - b_i)(b_i - c_i) \\ &\quad + \sum_{i=1}^n (b_i - c_i)^2 \\ &= \sum_{i=1}^n (a_i - c_i)^2 = [d(a, c)]^2. \end{aligned}$$

This implies

$$d(a, b) + d(b, c) \geq d(a, c).$$

Since $d(b, c) = d(c, b)$ holds obviously, d satisfies (M1). On the other hand, d clearly satisfies (M2) and (M5). Hence d is a metric in $X = R^n$. One can easily verify that the topology τ defined by the metric coincides with the product topology of R^n as the n -th topological power of the real line R . This metric space R^n will still be called the *Euclidean n -space* and its metric d will be referred to as the *Euclidean metric* in R^n .

(5) *The real Hilbert space H .* Let H denote the set of all sequences

$$x = (x_1, x_2, x_3, \dots)$$

of real numbers such that $\sum x_i^2$ is a convergent series. The real number x_i is called the i -th *coordinate* of the point x . Consider any two points a and b of H with their i -th coordinates denoted by a_i and b_i respectively. For any two real numbers α and β , the Schwarz inequality

$$\sum_{i=m}^n (\alpha a_i + \beta b_i)^2 \leq \alpha^2 \sum_{i=m}^n a_i^2 + 2|\alpha\beta| \left(\sqrt{\sum_{i=m}^n a_i^2} \right) \left(\sqrt{\sum_{i=m}^n b_i^2} \right) + \beta^2 \sum_{i=m}^n b_i^2$$

for any two positive integers m and n with $m < n$ implies that the sequence

$$\alpha a + \beta b = (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \dots)$$

is a point in H . In particular, if we take $\alpha = 1$ and $\beta = -1$, it follows that $\sum (a_i - b_i)^2$ is a convergent series. Hence we may define a function $d: H^2 \rightarrow R$ by taking

$$d(a, b) = \sqrt{\sum_{i=1}^{\infty} (a_i - b_i)^2}$$

for all points a and b of H . For any three points a, b, c of H , we have

$$d(a, c) + d(b, c) \geq \sqrt{\sum_{i=1}^n (a_i - c_i)^2} + \sqrt{\sum_{i=1}^n (b_i - c_i)^2} \geq \sqrt{\sum_{i=1}^n (a_i - b_i)^2}$$

because of the condition (M1) for the Euclidean metric in R^n . Since this is true for all positive integers n , we obtain

$$d(a, c) + d(b, c) \geq d(a, b).$$

Hence d satisfies (M1). Since d clearly satisfies (M2) and (M5), it follows that d is a metric in H . This metric space H is called the *real Hilbert space*.

(6) *The space $C(X)$.* Consider the set $\Phi = C(X)$ of all bounded continuous real-valued functions defined on any given topological space X . Let f and g denote any two members of $\Phi = C(X)$. For any two real numbers α and β , the function $\alpha f + \beta g: X \rightarrow R$ defined by

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$$

for every $x \in X$ is clearly bounded and continuous. In particular, $f - g$ is bounded. Hence we may define a function $d: \Phi^2 \rightarrow R$ by taking

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

for all members f and g of $\Phi = C(X)$. For any three members f, g, h of Φ and any point x in X , we have

$$d(f, h) + d(g, h) \geq |f(x) - h(x)| + |g(x) - h(x)| \geq |f(x) - g(x)|$$

because of the condition (M1) for the usual metric in R . Since this is true for all points $x \in X$, we obtain

$$d(f, h) + d(g, h) \geq d(f, g).$$

Hence d satisfies (M1). Since d clearly satisfies (M2) and (M5), it follows that d is a metric in $C(X)$. This metric space $C(X)$ is called the *space of all bounded continuous real-valued functions on the space X* .

Now let X denote an arbitrarily given pseudo-metric space. In the remainder of this section, we will study a few properties and consequences of the pseudo-metric $d: X^2 \rightarrow R$ of X .

THEOREM 2.2. *The pseudo-metric $d: X^2 \rightarrow R$ of a pseudo-metric space X is continuous.*

Proof. Let r be any real number. It suffices to prove that the sets

$$A = \{(x, y) \in X^2 \mid d(x, y) < r\}$$

$$B = \{(x, y) \in X^2 \mid d(x, y) > r\}$$

are open in the topological square X^2 of the space X .

To prove that A is open, let (a, b) denote any point in A . Then we have $d(a, b) = q < r$. Take $p = \frac{1}{2}(r - q)$ and consider the open set

$$U = N_p(a) \times N_p(b)$$

of X^2 . Then $(x, y) \in U$ implies that

$$\begin{aligned} d(x, y) &\leq d(x, a) + d(a, b) + d(b, y) \\ &< p + q + p = 2p + q = r. \end{aligned}$$

Hence we obtain $(a, b) \in U \subset A$. This proves that A is open.

To prove that B is open, let (a, b) denote any point in B . Then we have $d(a, b) = q > r$. Take $p = \frac{1}{2}(q - r)$ and consider the open set

$$U = N_p(a) \times N_p(b)$$

of X^2 . Then $(x, y) \in U$ implies that

$$d(a, b) \leq d(a, x) + d(x, y) + d(y, b).$$

Hence we obtain

$$\begin{aligned} d(x, y) &\geq d(a, b) - d(a, x) - d(b, y) \\ &> q - p - p = q - 2p = r. \end{aligned}$$

Hence we obtain $(a, b) \in U \subset B$. This proves that B is open. $\parallel$

For any two points a and b of X , the real number $d(a, b)$ will be called the *distance* between the points a and b with respect to the given pseudo-metric d , although this term is traditionally used only in the case where d is a metric.

Let A and B denote any two nonempty subsets of X . Then the *distance* $d(A, B)$ between the sets A and B is defined by

$$d(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\}.$$

In particular, if one of the two sets, say A , consists of a single point a , then $d(A, B)$ is simply denoted by $d(a, B)$ and is called the *distance from the point a to the set B* .

PROPOSITION 2.3. *For any given subset E of a pseudo-metric space X , the function $f: X \rightarrow R$ defined by*

$$f(x) = d(x, E)$$

for every $x \in X$ is continuous.

Proof. Let r denote any real number. It suffices to prove that the sets

$$A = \{x \in X \mid f(x) < r\}$$

$$B = \{x \in X \mid f(x) > r\}$$

are open in the space X .

To prove that A is open, let a denote any point in A . Then we have

$$d(a, E) = f(a) = q < r.$$

Take $p = \frac{1}{2}(r - q)$. Since $d(a, E) = q$, there exists a point $e \in E$ with $d(a, e) < p + q$. Consider the open set $U = N_p(a)$ of X . Then $x \in U$ implies that

$$f(x) = d(x, E) \leq d(x, e) \leq d(x, a) + d(e, a) < p + p + q = r.$$

Hence we obtain $a \in U \subset A$. This proves that A is open.

To prove that B is open, let b denote any point in B . Then we have

$$d(b, E) = f(b) = q > r.$$

Take $p = \frac{1}{2}(q - r)$ and consider the open set $V = N_p(b)$ of X . Then $x \in V$ and $e \in E$ imply

$$d(x, e) \geq d(b, e) - d(b, x) \geq d(b, E) - d(b, x) > q - p.$$

Since this is true for every $e \in E$, it follows that

$$f(x) = d(x, E) \geq q - p > r.$$

Hence we obtain $b \in V \subset B$. This proves that B is open. $\parallel$

PROPOSITION 2.4. *For any subset E of a pseudo-metric space X , we have*

$$\text{Cl}(E) = \{x \in X \mid d(x, E) = 0\}.$$

Proof. Because of (2.3), the set

$$F = \{x \in X \mid d(x, E) = 0\}$$

is a closed set of X . Since clearly $E \subset F$, we have $\text{Cl}(E) \subset F$.

On the other hand, let $x \in F$. Let U denote an arbitrary neighborhood of x in X . Since d defines the topology of X , there exists a positive real number r with $N_r(x) \subset U$. Because of

$$d(x, E) = 0 < r,$$

there exists a point $y \in E$ with $d(x, y) < r$. Hence we obtain $y \in U$. This implies $x \in \text{Cl}(E)$. Therefore, we have $F \subset \text{Cl}(E)$. $\parallel$

Now let us consider any locally finite open cover $\mathcal{C}$ of a pseudo-metric space X . Since $\mathcal{C}$ is locally finite and $d(x, X \setminus U) = 0$ for every $x \notin U$, we may define a map. $S: X \rightarrow R$ by taking

$$S(x) = \sum_{U \in \mathcal{C}} d(x, X \setminus U)$$

for every $x \in X$. On the other hand, since $\mathcal{C}$ covers X and $d(x, X \setminus U) > 0$ iff $x \in U$, we have $S(x) > 0$ for every $x \in X$. Hence we may define for each open set U in the cover $\mathcal{C}$ a map

$$f_U: X \rightarrow I$$

by taking

$$f_U(x) = d(x, X \setminus U) / S(x)$$

for every $x \in X$. Thus we have constructed a family

$$\mathcal{F} = \{f_U \mid U \in \mathcal{C}\}$$

of maps from X into the unit interval I satisfying the following two conditions:

$$(P1) \quad \sum_{U \in \mathcal{C}} f_U(x) = 1 \quad \text{for every } x \in X,$$

$$(P2) \quad f_U(x) > 0 \quad \text{iff } x \in U.$$

By a *partition of unity* on a topological space X , we mean a family

$$\mathcal{P} = \{p_\mu \mid \mu \in M\}$$

of maps from X into the unit interval I such that all but a finite number of members of $\mathcal{P}$ vanish outside some neighborhood of each point of X and that

$$\sum_{\mu \in M} p_\mu(x) = 1$$

holds for every $x \in X$. The partition $\mathcal{P}$ of unity is said to be *subordinate* to a cover $\mathcal{C}$ of X iff, for each $\mu \in M$, the *support* $\sigma(p_\mu)$ of p_μ , defined by

$$\sigma(p_\mu) = \{x \in X \mid p_\mu(x) > 0\},$$

is contained in some member of $\mathcal{C}$.

Thus we have proved the following theorem.

THEOREM 2.5. *For every locally finite open cover $\mathcal{C}$ of a pseudo-metric space X , the pseudo-metric d of X defines a partition*

$$\mathcal{F} = \{f_U \mid U \in \mathcal{C}\}$$

of unity on X such that the support $\sigma(f_U)$ of the map f_U is precisely the open set $U \in \mathcal{C}$.

EXERCISES

2A. Let A and B denote any two sets in a pseudo-metric space (X, d) . Prove the following assertions:

(i) $d[\text{Cl}(A), \text{Cl}(B)] = d(A, B)$.

(ii) If A and B are compact, then there exist points $a \in A$ and $b \in B$ with $d(a, b) = d(A, B)$.

2B. Let (X, d) denote any bounded metric space and consider the family X^* of all closed subsets of X . For every $A \in X^*$ and every positive real number r , let

$$N_r(A) = \{x \in X \mid d(x, A) < r\}.$$

Then define a function $d^*: X^{*2} \rightarrow R$ by taking

$$d^*(A, B) = \inf\{r \mid A \subset N_r(B), B \subset N_r(A)\}$$

for all A and B in X^* . Prove that d^* is a metric in X^* . This metric d^* is called the *Hausdorff metric* in X^* induced by the given metric d in X . Note that $d^*(A, B)$ is not necessarily equal to the distance $d(A, B)$ between A and B .

- 2C. Prove that a net $\phi: D \rightarrow X$ in any pseudo-metric space (X, d) converges to a point x of X iff the real-valued net $\psi: D \rightarrow R$ defined by

$$\psi(n) = d[x, \phi(n)]$$

for every $n \in D$ converges to the real number 0.

- 2D. Prove that every locally finite open cover $\mathcal{C}$ of a normal space X is *shrinkable*, that is, for each $U \in \mathcal{C}$, there exists an open set U^* of X such that the closure $\text{Cl}(U^*)$ is contained in U and that $\{U^* \mid U \in \mathcal{C}\}$ covers X . By means of this fact, prove that, for any given locally finite open cover $\mathcal{C}$ of a normal space X , there exists a partition

$$\mathcal{P} = \{f_U \mid U \in \mathcal{C}\}$$

of unity on X such that the support $\sigma(f_U)$ of the function f_U is contained in the open set $U \in \mathcal{C}$.

3. METRIZABILITY

A topological space X is said to be *pseudo-metrizable* iff there exists a pseudo-metric

$$d: X^2 \rightarrow R$$

which defines the topology of the space X . In case d is a metric, then we say that the space X is *metrizable*.

For examples, the real line R and discrete spaces are metrizable, and indiscrete spaces are pseudo-metrizable.

PROPOSITION 3.1. *A topological space X is pseudo-metrizable (metrizable) iff there exists an imbedding*

$$h: X \rightarrow Y$$

of the space X into a pseudo-metric (metric) space Y .

Proof. Necessity. Assume that X is pseudo-metrizable. Then there exists a pseudo-metric d in X which defines the topology of X . Let Y denote this pseudo-metric space. Then the identity function $h: X \rightarrow Y$ is a homeomorphism and hence an imbedding.

Sufficiency. Consider any imbedding $h: X \rightarrow Y$ of the space X into a pseudo-metric space Y . Let $e: Y^2 \rightarrow R$ denote the pseudo-metric of Y . Define a function $d: X^2 \rightarrow R$ by taking

$$d(a, b) = e[h(a), h(b)]$$

for any two points a and b in X . Since h is an imbedding, one can easily verify that d is a pseudo-metric in X and defines the topology of X .

This proves the "pseudo-metrizable" part of (3.1). The "metrizable" part can be proved similarly. ||

COROLLARY 3.2. *Every subspace of a pseudo-metrizable (metrizable) space is pseudo-metrizable (metrizable).*

COROLLARY 3.3. *The unit interval I is metrizable.*

THEOREM 3.4. *The topological sum of a disjoint family of pseudo-metrizable (metrizable) spaces is pseudo-metrizable (metrizable).*

Proof. Consider an arbitrarily given disjoint family

$$\mathcal{F} = \{X_\mu \mid \mu \in M\}$$

of pseudo-metrizable spaces. Then, for each $\mu \in M$, there exists a pseudo-metric d_μ in X_μ which defines the topology of X_μ . By the proof of (1.5), we may assume that

$$d_\mu(a, b) \leq 1$$

holds for every $\mu \in M$ and all points a and b in X_μ .

Let X denote the topological sum of the family $\mathcal{F}$. Define a function

$$d: X^2 \rightarrow R$$

as follows. Let a and b denote any two points in X . If there exists a $\mu \in M$ such that X_μ contains both a and b , set $d(a, b) = d_\mu(a, b)$; otherwise, set $d(a, b) = 1$.

To verify (M1) for d , let a, b, c denote arbitrary points in X . If there exists a $\mu \in M$ such that X_μ contains the three points a, b, c , then we have

$$d(a, b) = d_\mu(a, b) \leq d_\mu(a, c) + d_\mu(b, c) = d(a, c) + d(b, c).$$

Otherwise, we have

$$d(a, b) \leq 1 \leq d(a, c) + d(b, c).$$

Hence d satisfies (M1). On the other hand, (M2) is clearly satisfied by d . Therefore, d is a pseudo-metric in X .

For each $\mu \in M$, X_μ is both open and closed in the topological sum X . Since $d \mid X_\mu^2 = d_\mu$, it follows that d defines the topology of X . This proves that X is pseudo-metrizable.

Finally, assume that X_μ is metrizable for every $\mu \in M$. Then each d_μ is a metric. From the definition of d , it is clear that d is also a metric. This proves that X is metrizable. ||

THEOREM 3.5. *The topological product of a countable family of pseudo-metrizable (metrizable) spaces is pseudo-metrizable (metrizable).*

Proof. First, let us prove that the topological product

$$X = X_1 \times X_2 \times \cdots \times X_n$$

of a finite number of pseudo-metrizable spaces $X_1, X_2, \dots, X_n$ is pseudo-metrizable. For each $i = 1, 2, \dots, n$, let d_i denote a pseudo-metric in X_i which defines the topology of X_i . Define a function $d: X \rightarrow R$ by taking

$$d(a, b) = \sqrt{\sum_{i=1}^n [d_i(a_i, b_i)]^2}$$

for any two points $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ of X . One can verify that d is a pseudo-metric in X and defines the topology of X . This proves that X is pseudo-metrizable.

Next let us prove that the topological product X of a sequence

$$X_1, X_2, \dots, X_n, \dots$$

of pseudo-metrizable spaces is also pseudo-metrizable. For each $i = 1, 2, \dots$, let d_i denote a pseudo-metric in X_i which defines the topology of X_i . By the proof of (1.5), we may assume

$$d_i(a_i, b_i) \leq 1$$

holds for every positive integer i and all points a_i and b_i in X_i . Define a function $d: X \rightarrow R$ by taking

$$d(a, b) = \sum_{i=1}^{\infty} 2^{-i} d_i(a_i, b_i)$$

for any two points $a = (a_1, a_2, \dots)$ and $b = (b_1, b_2, \dots)$ of X . One can verify that d is a pseudo-metric in X and defines the topology of X . This proves that X is pseudo-metrizable.

Finally, assume that the spaces X_i in the preceding two cases are metrizable. Then each d_i is a metric. From the definitions of d in both cases, it is clear that d is also a metric. Hence X is metrizable. $\square$

COROLLARY 3.6. *The Euclidean n -space R^n , the unit n -cube I^n , and the Hilbert cube I^N are metrizable.*

In fact, there is a standard imbedding

$$h: I^N \rightarrow H$$

of the Hilbert cube I^N into the Hilbert space H defined as follows: For every point $\xi: N \rightarrow I$ in the Hilbert cube I^N , the i -th coordinate of the

point $h(\xi) \in H$ is the real number $i^{-1}\xi(i)$ for every $i \in N$. Hence we may also define the *Hilbert cube* as the subset $K = h(I^N)$ of the Hilbert space H which consists of all sequences

$$x = (x_1, x_2, x_3, \dots)$$

of real numbers satisfying the condition

$$0 \leq x_i \leq i^{-1}$$

for every $i \in N$.

The following theorem is an immediate consequence of (3.1), (3.6), and the Urysohn imbedding theorem (III, 6.6).

THEOREM 3.7. (URYSOHN METRIZATION THEOREM). *Every regular Fréchet space satisfying the second axiom of countability is metrizable.*

The remainder of the section is devoted to metrization theorems for spaces not necessarily satisfying the second axiom of countability.

A family $\mathcal{F}$ of subsets of a topological space X is said to be σ -locally finite iff it is the union of a countable number of locally finite subfamilies.

LEMMA 3.8. *Every regular space whose topology has a σ -locally finite basis is normal.*

Proof. Let X denote a regular space whose topology τ has a basis β which is the union of a sequence

$$\{\beta_n \mid n = 1, 2, \dots\}$$

of locally finite subfamilies β_n of β .

To prove that the space X is normal, let A and B denote any two disjoint closed subsets of X . Since X is regular, there exists for each point $a \in A$ a basic open set $M_a \in \beta$ satisfying

$$a \in M_a \subset \text{Cl}(M_a) \subset X \setminus B.$$

Similarly, for each $b \in B$, there exists a basic open set $N_b \in \beta$ satisfying

$$b \in N_b \subset \text{Cl}(N_b) \subset X \setminus A.$$

Thus we obtain open covers $\mathcal{U}$ and $\mathcal{V}$ of the subsets A and B defined by

$$\mathcal{U} = \{U \in \beta \mid U = M_a \text{ for some } a \in A\},$$

$$\mathcal{V} = \{V \in \beta \mid V = N_b \text{ for some } b \in B\}.$$

For each positive integer n , define

$$\mathcal{U}_n = \mathcal{U} \cap \beta_n, \quad \mathcal{V}_n = \mathcal{V} \cap \beta_n.$$

Then $\mathcal{U}_n$ and $\mathcal{V}_n$ are locally finite. Let

$$U_n = \bigcup \{U \mid U \in \mathcal{U}_n\}, \quad V_n = \bigcup \{V \mid V \in \mathcal{V}_n\}.$$

Since $\mathcal{U}_n$ is locally finite, it follows that

$$\text{Cl}(U_n) = \bigcup \{\text{Cl}(U) \mid U \in \mathcal{U}_n\} \subset X \setminus B.$$

Similarly, we have

$$\text{Cl}(V_n) = \bigcup \{\text{Cl}(V) \mid V \in \mathcal{V}_n\} \subset X \setminus A.$$

For each positive integer n , define open sets

$$P_n = U_n \setminus \bigcup \{\text{Cl}(V_i) \mid i \leq n\},$$

$$Q_n = V_n \setminus \bigcup \{\text{Cl}(U_i) \mid i \leq n\}.$$

Since $P_i \cap V_j$ is empty whenever $i \geq j$, it follows that $P_i \cap Q_j$ is empty whenever $i \geq j$. Similarly, since $U_i \cap Q_j$ is empty whenever $i \leq j$, it follows that $P_i \cap Q_j$ is empty whenever $i \leq j$. Consequently, we have

$$P_i \cap Q_j = \emptyset$$

for all positive integers i and j .

Finally, define open sets P and Q of X by

$$P = \bigcup \{P_n \mid n \in \mathbb{N}\}, \quad Q = \bigcup \{Q_n \mid n \in \mathbb{N}\}.$$

Then we have

$$A \subset P, \quad B \subset Q, \quad P \cap Q = \emptyset.$$

This proves that X is normal. $\square$

COROLLARY 3.9. *Every regular space which satisfies the second axiom of countability is normal.*

LEMMA 3.10. *Every regular space whose topology has a σ -locally finite basis is pseudo-metrizable.*

Proof. Let X denote any regular space whose topology τ has a basis β which is the union of a sequence

$$\{\beta_n \mid n = 1, 2, \dots\}$$

of locally finite subfamilies β_n of β .

Consider the Cartesian square $\Lambda = \mathbb{N}^2$ of the set $\mathbb{N}$ of all positive integers. Let $\lambda = (m, n)$ denote any point of Λ . For each $U \in \beta_m$, let

$$U^* = \bigcup \{V \in \beta_n \mid \text{Cl}(V) \subset U\}.$$

Since β_n is locally finite, it follows that

$$\text{Cl}(U^*) \subset U.$$

According to (3.8) and Urysohn's lemma (III, 1.6), there exists a map $f_U: X \rightarrow I$ satisfying

$$f_U(X \setminus U) = 0, \quad f_U[\text{Cl}(U^*)] = 1.$$

Then define a function $d_\lambda: X^2 \rightarrow R$ by taking

$$d_\lambda(a, b) = \sum_{U \in \beta_m} |f_U(a) - f_U(b)|$$

for any two points a and b of X . Since β_m is locally finite, the summation is finite and d_λ is continuous. On the other hand, d_λ obviously satisfies (M1) and (M2). Hence, for each $\lambda \in \Lambda$, d_λ is a continuous pseudo-metric in the space X .

For each $\lambda \in \Lambda$, the pseudo-metric d_λ makes the set X a pseudo-metric space which will be denoted by X_λ . Since d_λ is continuous, it follows that the identity function $j_\lambda: X \rightarrow X_\lambda$ is continuous. Thus we obtain an indexed family

$$\mathcal{F} = \{j_\lambda: X \rightarrow X_\lambda \mid \lambda \in \Lambda\}$$

of surjective maps from the given space onto the pseudo-metric spaces X_λ .

This family $\mathcal{F}$ obviously can distinguish points of X . To prove that $\mathcal{F}$ can also distinguish points from closed sets of X , let us consider an arbitrary closed set A of X and any point p in $X \setminus A$. Since β is a basis of the topology τ of X and $X \setminus A$ is open, there exists a $U \in \beta$ with $p \in U \subset X \setminus A$. Since X is a regular space, there exists a $V \in \beta$ satisfying $p \in V$ and $\text{Cl}(V) \subset U$. Choose positive integers m and n with $U \in \beta_m$ and $V \in \beta_n$. Then we have $\lambda = (m, n) \in \Lambda$ and

$$d_\lambda(p, A) \geq 1.$$

This implies that $j_\lambda(p)$ is not in the closure of $j_\lambda(A)$. Therefore, the family $\mathcal{F}$ can distinguish points from closed sets of X .

According to (III, 6.3), the Cartesian product

$$j: X \rightarrow Y$$

of the family $\mathcal{F}$ is an imbedding of the given space X into the topological product Y of the families $\{X_\lambda \mid \lambda \in \Lambda\}$ of pseudo-metrizable spaces. Since Λ is countable, it follows from (3.5) that Y is pseudo-metrizable. By (3.1), this implies that X is pseudo-metrizable. ||

A family $\mathcal{F}$ of subsets of a topological space X is said to be *discrete* iff every point of X has a neighborhood which meets at most one member of $\mathcal{F}$. A family $\mathcal{F}$ is said to be σ -*discrete* iff it is the union of a countable number of discrete subfamilies.

LEMMA 3.11. *Every open cover of a pseudo-metrizable space has a σ -discrete open refinement.*

Proof. Let $\mathcal{C}$ denote an arbitrary open cover of a pseudo-metrizable space X . Choose a pseudo-metric d in X which defines the topology of X .

For every member U of $\mathcal{C}$ and every positive integer n , let U_n denote the subset of U defined by

$$U_n = \{x \in U \mid d(x, X \setminus U) \geq 2^{-n}\}.$$

Then we have $U_n \subset U_{n+1}$ and we can easily verify

$$d(U_n, X \setminus U_{n+1}) \geq 2^{-n} - 2^{-n-1} = 2^{-n-1}$$

for every $n \in N$.

Next we order the family $\mathcal{C}$ by a relation $<$. For each $U \in \mathcal{C}$ and every positive integer n , let

$$U'_n = U_n \setminus \bigcup \{V_{n+1} \mid V \in \mathcal{C} \text{ and } V < U\}.$$

Let U and V denote any two members of $\mathcal{C}$. If $U < V$, we have $V'_n \subset X \setminus U_{n+1}$; if $V < U$, we have $U'_n \subset X \setminus V_{n+1}$. Hence, in either case, we have

$$d(U'_n, V'_n) \geq 2^{-n-1}.$$

Now, for each $U \in \mathcal{C}$ and each $n \in N$, define an open set U_n^* of X by

$$U_n^* = \{x \in U \mid d(x, U'_n) < 2^{-n-3}\}.$$

By means of (M1), one can easily verify that

$$d(U_n^*, V_n^*) \geq 2^{-n-2}$$

holds for any two distinct members U and V of the cover $\mathcal{C}$. Hence, for each $n \in N$, the family

$$\mathcal{B}_n = \{U_n^* \mid U \in \mathcal{C}\}$$

of open sets of X is discrete and the family

$$\mathcal{B} = \bigcup \{\mathcal{B}_n \mid n \in N\}$$

is σ -discrete.

It remains to prove that $\mathcal{B}$ is a refinement of the cover $\mathcal{C}$. For this purpose, let x denote an arbitrary point X . If U is the first member of the cover $\mathcal{C}$ which contains x , then x belongs to U_n^* for some $n \in N$. Hence $\mathcal{B}$ is an open cover of X . Since $U_n^* \subset U$ holds for every $U \in \mathcal{C}$ and every $n \in N$, $\mathcal{B}$ is a refinement of $\mathcal{C}$. ||

THEOREM 3.12. *For an arbitrary regular space X , the following three statements are equivalent:*

- (a) *The topology of X has a σ -discrete basis.*
- (b) *The topology of X has a σ -locally finite basis.*
- (c) *The space X is pseudo-metrizable.*

Proof. (a) $\Rightarrow$ (b) is obvious. (b) $\Rightarrow$ (c) is the lemma (3.10). (c) $\Rightarrow$ (a) is a consequence of the lemma (3.11) as follows. Let (X, d) denote any pseudo-metric space. For each $n \in \mathbb{N}$, select a σ -discrete open refinement β_n of the open cover of X consisting of all open balls of radius 2^{-n} . Then

$$\beta = \bigcup \{\beta_n \mid n \in \mathbb{N}\}$$

is a σ -discrete basis of the topology of the space X . $\parallel$

The following metrization theorem is a direct consequence of (3.12) and (2.1).

THEOREM 3.13. *For an arbitrary regular Fréchet space X , the following three statements are equivalent:*

- (a) *The topology of X has a σ -discrete basis.*
- (b) *The topology of X has a σ -locally finite basis.*
- (c) *The space X is metrizable.*

The equivalence (a) $\Leftrightarrow$ (c) is called the Bing metrization theorem, [Bing 1], and the equivalence (b) $\Leftrightarrow$ (c) is known as the Nagata-Smirnov metrization theorem, [Nagata 1, Smirnov 1].

EXERCISES

- 3A. Prove that every regular Lindelöf space is normal.
- 3B. A topological space X is said to be *locally metrizable* iff every point of X has a metrizable neighborhood. Prove that every locally metrizable paracompact Hausdorff space is metrizable. [Smirnov 2].
- 3C. Prove that a topological space X is metrizable iff at every point $p \in X$ there is a sequence $\beta(p) = \{W_n(p) \mid n \in \mathbb{N}\}$ of open neighborhoods such that:

- (a) $W_{n+1}(p) \subset W_n(p)$ for each $n \geq 1$.
- (b) $\bigcap_n W_n(p) = \{p\}$.
- (c) $\beta(p)$ is a local basis at p .

- (d) For each $p \in X$ and each $n \geq 1$, there is an $m \geq 1$ such that $W_m(q) \subset W_n(p)$ whenever $W_m(q)$ meets $W_m(p)$. [Frink 1].

3D. Prove that the image Y of a closed identification $f: X \rightarrow Y$ of a metrizable space X is metrizable iff the inverse image $f^{-1}(y)$ has a compact boundary in X for every $y \in Y$. [Morita and Hanai 1, Stone 2].

4. TOPOLOGICAL PROPERTIES

The present section is devoted to the topological properties of metric and pseudo-metric spaces, in other words, properties of metrizable and pseudo-metrizable spaces.

THEOREM 4.1. *Every pseudo-metrizable space is normal.*

Proof. Let X be any pseudo-metrizable space and $d: X^2 \rightarrow R$ be a pseudo-metric which defines the topology of X . To prove that X is normal, consider any two disjoint closed sets A and B in X . Let

$$U = \{x \in X \mid d(x, A) < d(x, B)\},$$

$$V = \{x \in X \mid d(x, A) > d(x, B)\}.$$

By (2.3), U and V are open sets of X . One can easily verify that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. Hence X is normal. $\parallel$

The following corollary is a direct consequence of (4.1), (2.1), and (III, 1.7).

COROLLARY 4.2. *Every metrizable space is a normal Hausdorff space.*

A topological space X is said to be *completely normal* iff every subspace of X is normal. The following proposition is an immediate consequence of (3.2) and (4.1).

PROPOSITION 4.3. *Every pseudo-metrizable space is completely normal.*

A subset E of a topological space X is said to be a G_δ iff it is the intersection of the members of a countable family of open sets of X . A topological space X is said to be *perfectly normal* iff X is normal and every closed subset of X is a G_δ .

PROPOSITION 4.4. *Every pseudo-metrizable space is perfectly normal.*

Proof. Let X be any pseudo-metrizable space and $d: X^2 \rightarrow R$ a pseudo-metric which defines the topology of X . Because of (4.1), it remains to prove that every closed subset of X is a G_δ . For this purpose,

let A denote an arbitrary closed subset of X . For every positive integer n , let

$$U_n = \{x \in X \mid d(x, A) < 2^{-n}\}.$$

According to (2.3), U_n is an open set of X . Since A is closed, it follows from (2.3) that

$$\bigcap \{U_n \mid n \in \mathbb{N}\} = A.$$

Hence A is a G_δ and X is perfectly normal. $\parallel$

Since a pseudo-metrizable space is not necessarily a Fréchet space, the following proposition is not a consequence of (4.1).

PROPOSITION 4.5. *Every pseudo-metrizable space is completely regular.*

Proof. Let X be any pseudo-metrizable space and $d: X^2 \rightarrow R$ be a pseudo-metric which defines the topology of X . To prove the complete regularity of X , consider an arbitrary point p in X and any open neighborhood U of p in X . By (2.4), we have

$$d(x, p) + d(x, X \setminus U) > 0$$

for every $x \in X$ since $X \setminus U$ is closed. Hence we may define a map $\chi: X \rightarrow I$ by taking

$$\chi(x) = \frac{d(x, p)}{d(x, p) + d(x, X \setminus U)}$$

for each $x \in X$. Since clearly we have $\chi(p) = 0$ and $\chi(X \setminus U) = 1$, X is completely regular. $\parallel$

THEOREM 4.6. *Every pseudo-metrizable space satisfies the first axiom of countability.*

Proof. Let p be an arbitrary point in a pseudo-metrizable space X and $d: X^2 \rightarrow R$ denote a pseudo-metric which defines the topology of X . For every positive rational number r , consider the r -neighborhood $N_r(p)$ of p in X . The family $\beta = \{N_r(p)\}$ is obviously a countable local basis at p . $\parallel$

Metrizable spaces may fail to satisfy the second axiom of countability since discrete spaces are metrizable.

THEOREM 4.7. *A pseudo-metrizable space X satisfies the second axiom of countability iff X is separable.*

Proof: Necessity. Because of (II, 2.9), a space X which satisfies the second axiom of countability must be separable.

Sufficiency. Assume that X is a separable pseudo-metrizable space. Let $d: X^2 \rightarrow R$ denote a pseudo-metric which defines the topology of X . Since X is separable, there exists a countable dense set E in X . Let Q denote the set of all positive rational numbers. Then, the family $\beta = \{N_q(e) \mid e \in E, q \in Q\}$ of open sets of X is countable. It remains to prove that β is a basis of the topology of X . For this purpose, consider an arbitrary open set U in X and any point p in U . Since the pseudo-metric d defines the topology of X , there exists a positive real number t satisfying

$$N_t(p) \subset U.$$

Let s denote a rational number satisfying $0 < 3s < t$. Since E is dense in X , there exists a point

$$e \in E \cap N_s(p).$$

Let $V = N_{2s}(e)$. Then $p \in V$. It remains to prove $V \subset U$. For this purpose, let $x \in V$. Then we have $d(e, x) < 2s$. Hence

$$d(p, x) \leq d(p, e) + d(e, x) < s + 2s = 3s < t.$$

This implies $x \in N_t(p) \subset U$. It follows that

$$p \in V \subset U.$$

Since $V \in \beta$, this proves that β is a basis of the topology of X . ||

THEOREM 4.8. *Every pseudo-metrizable space is paracompact.*

Proof. Let X denote an arbitrary pseudo-metrizable space and $d: X^2 \rightarrow R$ be any pseudo-metric which defines the topology of X .

Consider an arbitrarily given open cover $\mathcal{C}$ of X . According to (3.11), the given cover $\mathcal{C}$ has a σ -discrete open refinement $\mathcal{B}$. Then $\mathcal{B}$ is the union of a sequence

$$\{\mathcal{B}_n \mid n = 1, 2, \dots\}$$

of discrete subfamilies $\mathcal{B}_n$ of $\mathcal{B}$.

For each positive integer n and every member V of $\mathcal{B}_n$, let

$$V^* = V \setminus \bigcup \{U \mid U \in \mathcal{B}_m \text{ for some } m < n\}.$$

For each positive integer n , let

$$\mathcal{A}_n = \{V^* \mid V \in \mathcal{B}_n\}.$$

Since $\mathcal{B}_n$ is discrete and $V^* \subset V$, it follows that $\mathcal{A}_n$ is also discrete. One can easily verify that the family

$$\mathcal{A} = \bigcup \{\mathcal{A}_n \mid n \in N\}$$

covers X and is a refinement of the given cover $\mathcal{C}$.

To prove that $\mathcal{A}$ is locally finite, consider an arbitrary point of x . Let n denote the first positive integer such that there exists a member U of $\mathcal{B}_n$ containing x . Then U is a neighborhood of the point x which is disjoint from every member of $\mathcal{A}_m$ with $m > n$. Since $\mathcal{A}_n$ is discrete for every $n \in N$, this implies that $\mathcal{A}$ is locally finite.

Thus we have proved that every open cover $\mathcal{C}$ of X has a locally finite refinement $\mathcal{A}$. Unfortunately, the members of $\mathcal{A}$ are not necessarily open sets. However, since a pseudo-metrizable space X is regular according to (4.5), the proof is completed by the following lemma. ||

LEMMA 4.9. (MICHAEL'S LEMMA). *For an arbitrary regular space X , the following three statements are equivalent:*

- (a) *Every open cover of X has a locally finite refinement.*
- (b) *Every open cover of X has a locally finite closed refinement.*
- (c) *X is paracompact.*

Proof. (a) $\Rightarrow$ (b). Let $\mathcal{A}$ denote any open cover of X . Since X is regular, there exists an open cover $\mathcal{B}$ of X such that the collection of closures of members of $\mathcal{B}$ is a refinement of $\mathcal{A}$. According to (a), the open cover $\mathcal{B}$ of X has a locally finite refinement $\mathcal{C}$. Let $\mathcal{D}$ denote the collection of closures of members of $\mathcal{C}$. Then $\mathcal{D}$ is a closed cover of X . By the choice of the cover $\mathcal{B}$, $\mathcal{D}$ is a refinement of the given open cover $\mathcal{A}$. Since $\mathcal{C}$ is locally finite, it follows from the properties of closures that $\mathcal{D}$ is also locally finite. Hence $\mathcal{D}$ is a locally finite closed refinement of $\mathcal{A}$.

(b) $\Rightarrow$ (c). Let $\mathcal{A}$ denote any open cover of X . It suffices to establish the existence of a locally finite open refinement of $\mathcal{A}$. For this purpose let $\mathcal{B}$ denote a locally finite refinement of $\mathcal{A}$. Since $\mathcal{B}$ is locally finite, there exists an open cover $\mathcal{C}$ of X such that every member of $\mathcal{C}$ meets only a finite number of members of $\mathcal{B}$. By (b), this open cover $\mathcal{C}$ of X has a locally finite closed refinement $\mathcal{D}$. For each $B \in \mathcal{B}$, let

$$B^* = X \setminus \bigcup \{D \in \mathcal{D} \mid B \cap D = \emptyset\}.$$

Then B^* is an open set containing B . For each $D \in \mathcal{D}$, D meets B^* iff D meets B . For each $B \in \mathcal{B}$, select an $A_B \in \mathcal{A}$ with $B \subset A_B$. Let

$$\mathcal{E} = \{B^* \cap A_B \mid B \in \mathcal{B}\}$$

Then $\mathcal{E}$ is an open refinement of the given open cover $\mathcal{A}$. Since each member of the locally finite cover $\mathcal{D}$ of X meets only a finite number of members of $\mathcal{E}$, it follows that $\mathcal{E}$ is locally finite. Hence $\mathcal{E}$ is a locally finite open refinement of the given open cover $\mathcal{A}$.

(c) $\Rightarrow$ (a). This implication is obvious. ||

EXERCISES

- 4A. Prove that the topological product of an uncountable number of unit intervals is not perfectly normal and hence not pseudo-metrizable.
- 4B. Prove that a topological space is completely regular iff it is imbeddable into a topological product of pseudo-metrizable spaces.
- 4C. Prove that a pseudo-metrizable space satisfies the second axiom of countability iff it is a Lindelöf space.
- 4D. Prove that every regular Lindelöf space is paracompact.
- 4E. Prove that the Hilbert space H is not locally compact. Hence a metrizable space may fail to be locally compact.
- 4F. Let X denote an arbitrary Fréchet space. Prove that the following three statements are equivalent:
- (a) X is regular and satisfies the second axiom of countability.
 - (b) X is imbeddable into the Hilbert cube I^N .
 - (c) X is metrizable and separable.
- 4G. A subset E of a topological space X is said to be an F_σ iff it is the union of the members of a countable family of closed sets of X . Verify that E is an F_σ iff $X \setminus E$ is a G_δ . Hence show that a normal space X is perfectly normal iff every open subset of X is an F_σ .

5. COMPACT SUBSETS

Consider an arbitrary pseudo-metric space X with

$$d: X^2 \rightarrow R$$

denoting its pseudo-metric. For any nonempty subset E of X , we define

$$\delta(E) = \sup \{d(a, b) \mid a \in E, b \in E\}$$

to be the *diameter* of the subset E . If $\delta(E)$ is finite, then we say that the subset E is *bounded*.

THEOREM 5.1. *Every compact subset of a pseudo-metric space is bounded.*

Proof. Let E denote any compact subset of a pseudo-metric space (X, d) . Consider the family

$$\mathcal{F} = \{N_1(e) \mid e \in E\}$$

of the open balls of X with unit radius and with the points of E as centers. Obviously, $\mathcal{F}$ covers the set E . Since E is compact, there exist a finite number of points of E , say $e_1, \dots, e_n$, such that

$$E \subset N_1(e_1) \cup N_1(e_2) \cup \dots \cup N_1(e_n).$$

Let k denote the real number

$$k = \max \{d(e_i, e_j) \mid 1 \leq i \leq n, 1 \leq j \leq n\}.$$

Then we will prove $\delta(E) \leq k + 2$. For this purpose, let a and b denote any two points of E . Since $\mathcal{F}$ covers E , there exist i and j with $a \in N_1(e_i)$ and $b \in N_1(e_j)$. Then it follows from the triangle inequality (M1) that

$$d(a, b) \leq d(e_i, a) + d(e_i, e_j) + d(e_j, b) < k + 2.$$

Since a and b are arbitrary points of E , this implies $\delta(E) \leq k + 2$. Hence E is bounded. $\parallel$

Applying (5.1) to the special case $E = X$, we obtain the following corollary.

COROLLARY 5.2. *If X is a compact pseudo-metrizable space, then every pseudo-metric $d: X^2 \rightarrow R$ which defines the topology of X is bounded.*

Applying to the special case where X is the Euclidean n -space R with the Euclidean metric, we obtain the following corollary of (5.1), (III, 2.5), and (III, 2.12).

COROLLARY 5.3. *A subset E of the Euclidean n -space R^n is compact iff it is bounded and closed.*

THEOREM 5.4. *If $\mathcal{F}$ is a family of open sets of a pseudo-metric space X covering a compact subset E of X , then there exists a positive real number r such that, for every $e \in E$, there is an open set $U_e \in \mathcal{F}$ containing the open ball $N_r(e)$ of X .*

Proof. Since E is compact and $\mathcal{F}$ covers E , there exists a finite number of members of $\mathcal{F}$, say

$$U_1, U_2, \dots, U_n,$$

such that

$$E \subset U_1 \cup U_2 \cup \dots \cup U_n.$$

For every integer $i = 1, 2, \dots, n$, define a function $f_i: X \rightarrow R$ by taking

$$f_i(x) = d(x, X \setminus U_i)$$

for every $x \in X$. Define a function $f: X \rightarrow R$ by taking

$$f(x) = \sum_{i=1}^n f_i(x)$$

for every $x \in X$. By (2.3), each f_i is continuous and hence so is f .

Let e denote any point of E . Then there is an integer i with $1 \leq i \leq n$ and $e \in U_i$. This implies

$$f(e) \geq f_i(e) > 0.$$

Hence $f(E)$ is a compact set of positive real numbers. It follows that there exists a positive real number q satisfying $f(e) \geq q$ for every $e \in E$.

Now let $r = q/n$. Let e denote an arbitrary point of E . Since

$$\sum_{i=1}^n f_i(e) = f(e) \geq nr,$$

there exists an integer i with $1 \leq i \leq n$ and $f_i(e) \geq r$. Hence we obtain

$$d(e, X \setminus U_i) = f_i(e) \geq r.$$

This implies $N_r(e) \subset U_i$. ||

Applying (5.4) to the special case where the family $\mathcal{F}$ consists of a single open set U , we obtain the following corollary.

COROLLARY 5.5. *If an open set U of a pseudo-metric space X contains a compact subset E of X , then we have*

$$d(E, X \setminus U) > 0$$

provided that both E and $X \setminus U$ are nonempty.

Applying (5.4) to the special case $E = X$, we obtain the following corollary.

COROLLARY 5.6. (LEBESGUE'S COVERING LEMMA). *For every open cover $\mathcal{C}$ of a compact pseudo-metric space X , there exists a positive real number r such that every subset S of X with $\delta(S) < r$ is contained in some member of $\mathcal{C}$.*

This positive real number r is called a *Lebesgue number* of the open cover $\mathcal{C}$.

A nonempty subset E of a pseudo-metric space X is said to be *totally bounded* (or *precompact*) iff, for every positive real number r , there exists a finite subset F_r of E satisfying

$$E \subset \bigcup \{N_r(x) \mid x \in F_r\}.$$

The following two propositions can be established as in the proof of (5.1) with the number 1 replaced by the real number r .

PROPOSITION 5.7. *Every compact subset of a pseudo-metric space is totally bounded.*

PROPOSITION 5.8. *Every totally bounded subset of a pseudo-metric space is bounded.*

PROPOSITION 5.9. *Every totally bounded pseudo-metric space is separable.*

Proof. According to the definition of total boundedness, there exists, for each positive integer n , a finite subset $F_{1/n}$ of X such that

$$X = \bigcup \{N_{1/n}(x) \mid x \in F_{1/n}\}.$$

Let

$$F = \bigcup \{F_{1/n} \mid n \in \mathbb{N}\}.$$

Then F is a countable subset of X .

It remains to prove that F is dense in X . For this purpose, let p denote an arbitrary point of X and let U be any neighborhood of p in X . Since d defines the topology of X , there exists a positive real number r satisfying $N_r(p) \subset U$. Choose an integer $n \geq 1/r$. There exists a point

$$x \in F_{1/n} \subset F$$

satisfying $p \in N_{1/n}(x)$. This implies

$$d(p, x) < 1/n \leq r.$$

Hence we obtain $x \in N_r(p) \subset U$. This proves that F is dense in X . $\parallel$

COROLLARY 5.10. *Every compact pseudo-metrizable space is separable.*

A topological space X is said to be *countably compact* iff every countable open cover of X has a finite subcover. Therefore, every compact space is countably compact. The converse is obviously true for Lindelöf spaces and hence also for spaces satisfying the second axiom of countability.

A topological space X is said to be *sequentially compact* iff every sequence in X has a convergent subsequence. Compactness and sequential compactness are independent concepts; in fact, there are compact spaces which fail to be sequentially compact and vice versa.

THEOREM 5.11. *For an arbitrary pseudo-metrizable space X , the following three statements are equivalent:*

- (a) X is compact.
- (b) X is countably compact.
- (c) X is sequentially compact.

Proof. The equivalence (a) $\Leftrightarrow$ (b) is a consequence of (4.7), (5.9), and the following lemma (5.12). The equivalence (b) $\Leftrightarrow$ (c) is a consequence of the following lemmas (5.13) and (5.14) together with (4.6). $\square$

LEMMA 5.12. *Every countably compact pseudo-metric space is totally bounded.*

Proof. Let X denote an arbitrary pseudo-metric space which is not totally bounded. We will prove that X is not countably compact.

Since X is not totally bounded, there exists a positive real number r such that no finite subset F of X satisfies

$$X = \bigcup \{N_r(x) \mid x \in F\}.$$

Because of this fact, one can construct inductively an infinite sequence

$$S = \{x_1, x_2, \dots, x_n, \dots\}$$

of distinct points of X such that

$$S \cap N_r(x_i) = \{x_i\}$$

holds for every positive integer i .

Consider the closed set H and the open set W of X defined by

$$H = X \setminus \bigcup \{N_r(x_i) \mid i \in N\},$$

$$W = \bigcup \{N_{r/2}(x) \mid x \in H\}.$$

Then the open balls $\{N_r(x_i) \mid i \in N\}$ and the open set W constitute a countable open cover $\mathcal{C}$ of the space X . This cover $\mathcal{C}$ has no finite subcover since each point x_n is not covered by the open set W nor by any of the open balls $N_r(x_i)$ with $i \neq n$. Hence X is not countably compact. $\square$

LEMMA 5.13. *For any topological space, sequential compactness implies countable compactness.*

Proof. Assume that a topological space X is not countably compact. Then, by definition, there exists an open cover

$$\mathcal{C} = \{U_n \mid n = 1, 2, \dots\}$$

of X which has no finite subcover. Then we can select a sequence

$$f: N \rightarrow X$$

such that

$$f(n) \notin U_1 \cup U_2 \cup \dots \cup U_n$$

for every integer $n \in N$.

To prove that X is not sequentially compact, it suffices to verify that the sequence f has no convergent subsequence. For this purpose,

let x denote any point of X . Since $\mathcal{C}$ covers X , there exists an $n \in N$ with $x \in U_n$. Since U_n is open, it is a neighborhood of x . By the choice of the sequence f , U_n can contain at most the points $f(i)$ with $i < n$. Hence no subsequence of f can converge to the point x . Since x is an arbitrary point of X , this implies that the sequence f has no convergent subsequence. $\parallel$

LEMMA 5.14. *For any topological space satisfying the first axiom of countability, countable compactness implies sequential compactness.*

Proof. Let X denote any topological space which satisfies the first axiom of countability and is countably compact. Consider an arbitrarily given sequence

$$f: N \rightarrow X.$$

First, let us prove by the method of contradiction that this sequence f must have a cluster point as defined in (III, § 3). For this purpose, let us assume that f has no cluster point. Then, for every point $x \in X$, there exists an open neighborhood U_x of x in X such that the inverse image $f^{-1}(U_x)$ is at most a finite subset of N . Let Φ denote the family of all finite subsets of N , including the empty set $\square$. For each $F \in \Phi$, let

$$U_F = \bigcup \{U_x \mid f^{-1}(U_x) = F\}.$$

Then the family

$$\mathcal{C} = \{U_F \mid F \in \Phi\}$$

is a countable open cover of the space X . Since $f^{-1}(U_F)$ is the finite subset of the infinite set N , $\mathcal{C}$ admits no finite subcover. This contradicts the countable compactness of X and hence proves the existence of a cluster point of the sequence f .

Next, let p denote a cluster point of the sequence f . We will construct a subsequence of f which converges to the point p . Since X satisfies the first axiom of countability, there exists a local basis at p consisting of the open sets

$$V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$$

of X . Construct an increasing function

$$h: N \rightarrow N$$

as follows. Select any integer from the infinite subset $f^{-1}(V_1)$ of N and denote it by $h(1)$. Let $n > 1$ and assume that $h(m)$ has been defined for all $m < n$. Since $f^{-1}(V_n)$ is infinite, there exists an integer $h(n) \in f^{-1}(V_n)$ satisfying

$$h(n) > h(n-1).$$

This completes the inductive construction of the increasing function h . Then the composition

$$g = f \circ h: N \rightarrow X$$

is, by definition, a subsequence of f . For any given $n \in N$, we have

$$g(m) = f[h(m)] \in V_m \subset V_n$$

for every $m \geq n$. This proves that g converges to the point p . Hence X is sequentially compact. $\parallel$

EXERCISES

5A. Let E be a nonempty compact subset and F be any nonempty subset of a pseudo-metric space (X, d) . Prove:

(a) There exists a point $e \in E$ with

$$d(e, F) = d(E, F).$$

(b) There exist points a and b in E with

$$d(a, b) = \delta(E).$$

5B. Let A , B , and E denote arbitrary subsets of a pseudo-metric space (X, d) . Prove:

(a) $\delta[\text{Cl}(E)] = \delta(E)$.

(b) $A \subset B$ implies $\delta(A) \leq \delta(B)$.

(c) $A \cap B \neq \emptyset$ implies $\delta(A \cup B) \leq \delta(A) + \delta(B)$.

(d) $x \in A$ and $y \in B$ imply $d(A, B) \leq d(x, y) \leq \delta(A \cup B)$.

(e) $\delta(A \cup B) \leq \delta(A) + d(A, B) + \delta(B)$.

5C. Prove that the Hilbert space H is separable. Hence a separable metrizable space may fail to be locally compact.

5D. Let E denote any subset of a given topological space X . A point $p \in X$ is said to be an ∞ -accumulation point of E in X iff $E \cap U$ is infinite for every neighborhood U of p in E . Every ∞ -accumulation point is an accumulation point in the sense of (II, § 2). Prove that the converse is true in case X is a Fréchet space.

- 5E. Prove that the following three statements are equivalent for any topological space X :
- (a) X is countably compact.
 - (b) Every infinite subset of X has an ∞ -accumulation point in X .
 - (c) Every sequence in X has a cluster point.
- 5F. Prove that every subspace of a totally bounded pseudo-metric space is totally bounded.

6. UNIFORMLY CONTINUOUS MAPS

Consider any two pseudo-metric spaces X and Y with

$$d_X: X^2 \rightarrow R, \quad d_Y: Y^2 \rightarrow R$$

denoting their pseudo-metrics.

PROPOSITION 6.1. *A function $f: X \rightarrow Y$ is continuous at a point $p \in X$ iff, for every positive real number ε , there exists a positive real number δ such that*

$$d_Y[f(p), f(x)] < \varepsilon$$

holds for every point $x \in X$ satisfying

$$d_X(p, x) < \delta.$$

Proof: Necessity. Assume that $f: X \rightarrow Y$ is continuous at the point $p \in X$ and let ε denote any given positive real number. Since the open ball $N_\varepsilon[f(p)]$ is a neighborhood of the point $f(p)$ in Y , it follows from the continuity of f at the point p that there exists a neighborhood U of p in X with $f(U) \subset N_\varepsilon[f(p)]$. Since d_X defines the topology of X , there exists a positive real number δ with $N_\delta(p) \subset U$. This implies

$$f[N_\delta(p)] \subset N_\varepsilon[f(p)].$$

Hence $d_Y[f(p), f(x)] < \varepsilon$ holds for every $x \in X$ satisfying $d_X(p, x) < \delta$.

Sufficiency. Assume that the condition is satisfied for every $\varepsilon > 0$. To prove the continuity of f at the point p , let V denote any neighborhood of the point $f(p)$ in Y . Since d_Y defines the topology of Y , there exists a positive real number ε with $N_\varepsilon[f(p)] \subset V$. According to our condition, there exists a positive real number δ such that $d_Y[f(p), f(x)] < \varepsilon$ holds for every point $x \in X$ satisfying $d_X(p, x) < \delta$. This implies

$$f[N_\delta(p)] \subset N_\varepsilon[f(p)] \subset V.$$

Since $N_\delta(p)$ is a neighborhood of p in X , f is continuous at the point p by definition. $\parallel$

Applying (6.1) to every point $p \in X$, we obtain the following corollary.

COROLLARY 6.2. *A function $f: X \rightarrow Y$ is continuous iff, for every point p in X and every positive real number ε , there exists a positive real number δ such that*

$$d_Y[f(p), f(x)] < \varepsilon$$

holds for every point $x \in X$ satisfying

$$d_X(p, x) < \delta.$$

The number δ in (6.2) depends not only on the number ε but also on the point p of X . For a given positive real number ε , it might be impossible to choose one $\delta > 0$ satisfying the condition for all points $p \in X$. This suggests the following definition.

A function $f: X \rightarrow Y$ is said to be *uniformly continuous* (with respect to the pseudo-metrics d_X and d_Y) iff, for every positive real number ε , there exists a positive real number δ such that

$$d_Y[f(p), f(q)] < \varepsilon$$

holds for all points p and q in X satisfying

$$d_X(p, q) < \delta.$$

Obviously, every uniformly continuous function $f: X \rightarrow Y$ is continuous. The converse is not true. For example, the continuous function $f: J \rightarrow R$ defined on the open unit interval $J = (0, 1)$ of real numbers by

$$f(x) = 1/x$$

for every $x \in J$ fails to be uniformly continuous with respect to usual metrics in J and R .

THEOREM 6.3. *If X is compact, then every continuous function $f: X \rightarrow Y$ is uniformly continuous.*

Proof. Let ε denote an arbitrary positive real number. For each point $y \in Y$, consider the open ball

$$V_y = N_{\varepsilon/2}(y)$$

in the space Y . Then the family

$$\mathcal{C} = \{f^{-1}(V_y) \mid y \in Y\}$$

is an open cover of the space X . Since X is compact, it follows from (5.6) that this cover $\mathcal{C}$ of X has a Lebesgue number δ which is a positive real number.

Let p and q denote any two points satisfying $d(p, q) < \delta$. Then it follows from (5.6) that the set $\{p, q\}$ is contained in some member of the cover $\mathcal{C}$. This implies the existence of a point $y \in Y$ with $f(p) \in V_y$ and $f(q) \in V_y$. Hence we obtain

$$d_Y[f(p), f(q)] \leq d_Y[y, f(p)] + d_Y[y, f(q)] < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves the uniform continuity of the function f . $\parallel$

In the remainder of the present section, we are concerned with a special kind of uniformly continuous maps defined as follows.

A function $f: X \rightarrow Y$ is said to be *submetric* iff

$$d_Y[f(p), f(q)] \leq d_X(p, q)$$

holds for all points p and q of the space X . In particular, a function $f: X \rightarrow Y$ is said to be *isometric* iff

$$d_Y[f(p), f(q)] = d_X(p, q)$$

holds for all points p and q of the space X .

Obviously, every submetric function $f: X \rightarrow Y$ is uniformly continuous; in fact, we have $\delta = \varepsilon$ for every $\varepsilon > 0$. The converse is not true. For example, the function $f: R \rightarrow R$ defined by

$$f(x) = 2x$$

for every $x \in R$ is uniformly continuous but not submetric.

LEMMA 6.4. *Surjective isometric functions are open maps.*

Proof. To prove that any surjective isometric function $f: X \rightarrow Y$ is an open map, it remains to show that the image

$$V = f(U)$$

of any open set U of X is an open set of Y . For this purpose, let q denote an arbitrary point in V . Then there exists a point $p \in U$ with $f(p) = q$. Since d_X defines the topology of X , there exists a positive real number r with $N_r(p) \subset U$. Let y denote any point in $N_r(q)$. Since f is surjective, there exists an $x \in X$ with $f(x) = y$. Then we have

$$d_X(p, x) = d_Y(q, y) < r.$$

This implies $x \in N_r(p) \subset U$ and, therefore, we obtain

$$y = f(x) \in f(U) = V.$$

This proves $N_r(q) \subset V$. Hence V is an open set of Y . $\parallel$

THEOREM 6.5. *If X is a metric space, then every isometric function $f: X \rightarrow Y$ is an imbedding of X into Y .*

Proof. To prove that f is injective, let a and b denote any two distinct points in X . Since X is a metric space, we have $d_X(a, b) > 0$. Since f is isometric, this implies $d_Y[f(a), f(b)] > 0$. Hence we must have $f(a) \neq f(b)$. This proves that f is injective. Then it follows from (6.4) that f is an imbedding of the metric space X into the pseudo-metric space Y . $\parallel$

EXAMPLES OF ISOMETRIC MAPS:

(1) Let X be any pseudo-metric space with $d: X^2 \rightarrow R$ denoting its pseudo-metric. Define an equivalence relation $\sim$ in X by taking $a \sim b$ iff $d(a, b) = 0$. Consider the quotient space

$$X^* = X/\sim$$

and the natural projection

$$p: X \rightarrow X^*.$$

Define a function $d^*: X^{*2} \rightarrow R$ by taking

$$d^*(\alpha, \beta) = d[p^{-1}(\alpha), p^{-1}(\beta)]$$

for all points α and β in X^* . By means of the triangle inequality (M1) and the equivalence $a \sim b \Leftrightarrow d(a, b) = 0$, one can easily verify that

$$d^*(\alpha, \beta) = d(a, b)$$

holds for every $a \in p^{-1}(\alpha)$ and $b \in p^{-1}(\beta)$. Then it follows that d^* is a metric in X^* and defines the quotient topology of X^* . The metric space (X^*, d^*) will be referred to as the *reduced metric space* of the given pseudo-metric space X . Since

$$d^*[p(a), p(b)] = d(a, b)$$

holds for all points a and b in X , the natural projection $p: X \rightarrow X^*$ is a surjective isometric map. If X is a metric space, then it is obvious that $X^* = X$ and p is the identity map.

(2) Let X be any pseudo-metric space with $d: X^2 \rightarrow R$ denoting its pseudo-metric. Consider the metric space $C(X)$ of all bounded continuous real-valued functions on X as defined in the example (6) of § 2.

To avoid possible ambiguity, we will denote the metric in $C(X)$ by the symbol $d^\#$. We will construct an isometric map

$$q: X \rightarrow C(X).$$

For this purpose, let us fix an arbitrary point z of X as our *base point*. For each point $x \in X$, define a function $f_x: X \rightarrow R$ by taking

$$f_x(t) = d(x, t) - d(z, t)$$

for every $t \in X$. By (2.2), f_x is continuous. By (M2), one can easily verify that

$$|f_x(t)| = |d(x, t) - d(z, t)| \leq d(z, x)$$

holds for all $t \in X$. Hence, f_x is bounded. It follows that the assignment $x \rightarrow f_x$ defines a function

$$q: X \rightarrow C(X).$$

It remains to prove that q is isometric. For this purpose, let a and b denote any two points of X . Then we have

$$\begin{aligned} d^\#[q(a), q(b)] &= \sup_{t \in X} |f_a(t) - f_b(t)| \\ &= \sup_{t \in X} |d(a, t) - d(b, t)|. \end{aligned}$$

By taking $t = b$, we obtain $d^\#[q(a), q(b)] \geq d(a, b)$. On the other hand, since $|d(a, t) - d(b, t)| \leq d(a, b)$ holds for every $t \in X$, we have $d^\#[q(a), q(b)] \leq d(a, b)$. Consequently, we obtain

$$d^\#[q(a), q(b)] = d(a, b)$$

for all points a and b of X . This proves that q is an isometric map. In case X is a metric space, it follows from (6.5) that q is an imbedding of X into $C(X)$ called the *canonical imbedding* with z as *base point*.

(3) For any pseudo-metric space (X, d) , consider the canonical imbedding

$$q: X \rightarrow C(X)$$

constructed in the preceding example. Let a and b denote two points in X satisfying $d(a, b) = 0$. Then it follows from the triangle inequality (M1) that

$$d(a, t) \leq d(a, b) + d(b, t) = d(b, t)$$

$$d(b, t) \leq d(a, b) + d(a, t) = d(a, t)$$

hold for all $t \in X$. Hence we obtain

$$d(a, t) = d(b, t)$$

for every $t \in X$. This implies

$$q(a) = f_a = f_b = q(b).$$

Therefore, q induces a unique function

$$q^*: X^* \rightarrow C(X)$$

from the reduced metric space X^* into $C(X)$ satisfying the relation

$$q^* \circ p = q$$

where $p: X \rightarrow X^*$ denotes the natural projection. Since

$$d^\# [q(a), q(b)] = d(a, b) = d^* [p(a), p(b)]$$

holds for all points a and b in X , it follows that q^* is isometric. Hence, by (6.5), q^* is an imbedding of X^* into $C(X)$. In case X is metric, then we have $X^* = X$ and $q^* = q$.

A property P of pseudo-metric spaces is said to be a *metric property* iff it is preserved by all bijective isometric maps. For example, boundedness, total boundedness, and the diameter of a pseudo-metric space are metric properties.

Every topological property of a pseudo-metric space is obviously a metric property. The converse is false, for the metric properties named above are not topological properties.

EXERCISES

- 6A. Prove that every isometric map $h: X \rightarrow X$ of a compact metric space is bijective by the following method or otherwise. Select a point $x \in X \setminus h(X)$ and define a sequence $f: N \rightarrow X$ by taking $f(1) = x$ and $f(n) = h[f(n-1)]$ for every $n > 1$. Prove that this sequence f has no convergent subsequence.
- 6B. Prove that an isometric map $f: X \rightarrow Y$ of a compact metric space X into a metric space Y is bijective iff there exists an isometric map $g: Y \rightarrow X$.
- 6C. A subset E of a topological space X is said to be *nowhere dense* iff every nonempty open set U of X contains a nonempty open set V satisfying $E \cap V = \emptyset$. Prove that a subset E of a topological space X is nowhere dense iff

$$\text{Int} [\text{Cl} (E)] = \emptyset.$$

6D. For an arbitrary subset E of a topological space X , prove the following assertions:

(a) If E is nowhere dense, then $X \setminus E$ is dense in X .

(b) If E is closed and dense in X , then $X \setminus E$ is nowhere dense.

6E. Prove that the function $f: H \rightarrow H$ of the real Hilbert space H defined by

$$f(x) = (0, x_1, x_2, \dots)$$

for every point $x = (x_1, x_2, \dots)$ of H is isometric and its image $f(H)$ is a nowhere dense subset of H .

6F. A sequence of functions $\{f_n: X \rightarrow Y \mid n \in N\}$ of a topological space X into a pseudo-metric space (Y, d) is said to be *uniformly convergent* to a function $f: X \rightarrow Y$ iff, for every positive real number ε , there exists a positive integer k such that

$$d[f_n(x), f(x)] < \varepsilon$$

holds for every $n > k$ and every $x \in X$. Prove that $f: X \rightarrow Y$ must be continuous in case there is a sequence of maps $\{f_n: X \rightarrow Y \mid n \in N\}$ which converges uniformly to f .

7. COMPLETENESS

Let X be any pseudo-metric space with $d: X^2 \rightarrow R$ denoting its pseudo-metric. By a *Cauchy sequence* (or *fundamental sequence*) in X , we mean a sequence

$$f: N \rightarrow X$$

such that, for every positive real number ε , there exists a positive integer n such that

$$d[f(i), f(j)] < \varepsilon$$

holds for all integers $i > n$ and $j > n$.

PROPOSITION 7.1. *Every convergent sequence $f: N \rightarrow X$ in a pseudo-metric space X is a Cauchy sequence.*

Proof. Assume that the sequence $f: N \rightarrow X$ converges to a point $p \in X$. To prove that f is a Cauchy sequence, consider an arbitrary positive real number ε . Since f converges to p , there exists a positive integer n such that

$$d[p, f(m)] < \varepsilon/2$$

holds for all integers $m > n$. Then we obtain

$$d[f(i), f(j)] \leq d[p, f(i)] + d[p, f(j)] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all integers $i > n$ and $j > n$. Hence f is a Cauchy sequence. $\parallel$

The converse of (7.1) is not always true. For example, in the open unit interval $J = (0, 1)$ of real numbers with the usual metric, the sequence $f: N \rightarrow J$ defined by

$$f(n) = 1/n$$

for every $n \in N$ is a Cauchy sequence but does not converge to some point in J . This suggests the following concept.

A pseudo-metric space X is said to be *complete* iff every Cauchy sequence in X converges to some point in X .

The following proposition is essentially the classical Cauchy theorem which is a direct consequence of the definition of real numbers.

PROPOSITION 7.2. *The real line R with its usual metric is complete.*

Completeness of pseudo-metric spaces is obviously a metric property. By the example given above, it follows that the open unit interval J with its usual metric is not complete. Since R and J are homeomorphic, completeness is not a topological property.

PROPOSITION 7.3. *Every closed subspace of a complete pseudo-metric space is complete.*

Proof. Let E denote a closed subspace of a complete pseudo-metric space (X, d) . The pseudo-metric of E is understood to be the restriction of d over E^2 . To prove the completeness of E , consider any Cauchy sequence $f: N \rightarrow E$ and the inclusion map $h: E \rightarrow X$. Then the composition

$$g = h \circ f: N \rightarrow X$$

is obviously a Cauchy sequence in X . Since X is complete, g converges to some point $p \in X$. Since

$$g(N) = f(N) \subset E$$

and since E is closed, it follows that p is in E . Hence f converges to the point $p \in E$. This proves the completeness of the subspace E . $\parallel$

PROPOSITION 7.4. *The Cartesian product*

$$X = X_1 \times \cdots \times X_n$$

of a finite number of complete pseudo-metric spaces $(X_1, d_1), \dots, (X_n, d_n)$ and the pseudo-metric $d: X^2 \rightarrow R$ defined by

$$d(a, b) = \sqrt{\sum_{i=1}^n [d(a_i, b_i)]^2}$$

for any two points $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ of X constitute a complete pseudo-metric space.

Proof. That d is a pseudo-metric in X and defines the product topology of X was established in the proof of (3.5). It remains to prove the completeness of the pseudo-metric space (X, d) .

For this purpose, let us consider any Cauchy sequence

$$f: N \rightarrow X$$

in the pseudo-metric space X . Since

$$d(a, b) \geq d_i(a_i, b_i)$$

holds for all points $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ of X and all integers $i = 1, \dots, n$, it follows that the composition

$$f_i = p_i \circ f: N \rightarrow X_i$$

of $f: N \rightarrow X$ and the natural projection $p_i: X \rightarrow X_i$ is a Cauchy sequence in X_i for every $i = 1, \dots, n$. Since (X_i, d_i) is complete, it follows that f_i converges to some point $x_i \in X_i$ for every $i = 1, \dots, n$.

It remains to prove that the sequence f converges to the point

$$x = (x_1, \dots, x_n) \in X.$$

For this purpose, consider any positive real number ε . Since f_i converges to x_i , there exists a positive integer m_i such that

$$d_i[x_i, f_i(k)] < \varepsilon/\sqrt{n}$$

holds for all integers $k > m_i$. Choose m_i for each $i = 1, \dots, n$ and let

$$m = \max \{m_1, \dots, m_n\}.$$

Then we obtain

$$d[x, f(k)] = \sqrt{\sum_{i=1}^n \{d_i[x_i, f_i(k)]\}^2} < \varepsilon$$

for every integer $k > m$. This proves that f converges to x . $\parallel$

COROLLARY 7.5. *The Euclidean n -space R^n with its Euclidean metric is complete.*

PROPOSITION 7.6. *The real Hilbert space H is complete.*

Proof. Consider an arbitrary Cauchy sequence $f: N \rightarrow H$. For each positive integer n , let $p_n: H \rightarrow R$ denote the natural projection defined by $p_n(x) = x_n$ for every point $x = (x_1, x_2, \dots)$ of H . It follows from the definition of the metric d in H that the composition

$$f_n = p_n \circ f: N \rightarrow R$$

is a Cauchy sequence of real numbers. By (7.2), f_n converges to a real number r_n for every positive integer n . Thus we obtain a sequence of real numbers

$$r = (r_1, r_2, \dots).$$

We shall prove that r is in H and that the sequence f converges to r . For this purpose, consider an arbitrary positive real number ε . Since f is a Cauchy sequence, there exists a positive integer k such that

$$\sqrt{\sum_{i=1}^m |f_n(i) - f_n(j)|^2} \leq d[f(i), f(j)] < \varepsilon$$

holds for integers $i > k, j > k$ and every positive integer m . If we let i increase while m and j are held fixed, we obtain

$$\sqrt{\sum_{n=1}^m |r_n - f_n(j)|^2} \leq \varepsilon.$$

Since this is true for all positive integers m , it follows that

$$\sqrt{\sum_{n=1}^{\infty} |r_n - f_n(j)|^2} \leq \varepsilon$$

holds for every integer $j > k$. In case $r \in H$, then we have

$$d[r, f(j)] = \sqrt{\sum_{n=1}^{\infty} |r_n - f_n(j)|^2} \leq \varepsilon$$

for all $j > k$. This proves that the sequence f converges to r if r is in H .

It remains to prove $r \in H$. For this purpose, let us first observe that

$$\begin{aligned} r_n^2 &= [r_n - f_n(j) + f_n(j)]^2 \\ &= [r_n - f_n(j)]^2 + 2[r_n - f_n(j)][f_n(j)] + [f_n(j)]^2 \\ &\leq 2[r_n - f_n(j)]^2 + 2[f_n(j)]^2 \end{aligned}$$

holds because of the elementary inequality

$$2ab \leq a^2 + b^2$$

for real numbers a and b . Hence we obtain

$$\sum_{n=1}^m r_n^2 \leq 2 \sum_{n=1}^m [r_n - f_n(j)]^2 + 2 \sum_{n=1}^m [f_n(j)]^2 \leq 2\varepsilon + 2 \sum_{n=1}^m [f_n(j)]^2$$

for integers $m > 0$ and $j > k$. Since $f(j)$ is a point of H , the infinite series $\sum [f_n(j)]^2$ is convergent. Consequently, $\sum r_n^2$ is also convergent and hence r is in H . $\parallel$

PROPOSITION 7.7. *The metric space $C(X)$ of all bounded continuous real-valued functions on a topological space X is complete.*

Proof. Consider an arbitrary Cauchy sequence

$$f: N \rightarrow C(X).$$

For each point $x \in X$, define a sequence

$$f_x: N \rightarrow R$$

of real numbers by taking

$$f_x(n) = [f(n)](x)$$

for every $n \in N$.

Let ε denote an arbitrary positive real number. Since f is a Cauchy sequence, there exists a positive integer k such that

$$d[f(i), f(j)] < \varepsilon$$

holds for all integers $i > k$ and $j > k$. Then it follows from the definition of the metric d of $C(X)$ that

$$|f_x(i) - f_x(j)| \leq d[f(i), f(j)] < \varepsilon$$

holds for all $x \in X$, $i > k$, and $j > k$. Hence f_x is a Cauchy sequence of real numbers and hence converges to a real number r_x . The assignment $x \rightarrow r_x$ defines a real-valued function $r: X \rightarrow R$.

In the following inequalities

$$f_x(j) - \varepsilon < f_x(i) < f_x(j) + \varepsilon,$$

let i increase while j and x are held fixed. Since f_x converges to the number r_x , we obtain

$$f_x(j) - \varepsilon \leq r_x \leq f_x(j) + \varepsilon$$

for all $x \in X$ and $j > k$. Since $r(x) = r_x$ and $[f(j)](x) = f_x(j)$, we obtain

$$|[f(j)](x) - r(x)| \leq \varepsilon$$

for all $x \in X$ and $j > k$. In case $r \in C(X)$, then we have

$$d[f(j), r] = \sup_{x \in X} |[f(j)](x) - r(x)| \leq \varepsilon$$

for all $j > k$. This proves that the sequence f converges to r .

It remains to prove that the function $r: X \rightarrow R$ is bounded and continuous. To prove that r is bounded, let us fix an integer $j > k$. Since $f(j)$ is bounded, there exists a real number b such that

$$|[f(j)](x)| \leq b$$

holds for all $x \in X$. Hence we obtain

$$|r(x)| \leq |[f(j)](x)| + |[f(j)](x) - r(x)| \leq b + \varepsilon$$

for all $x \in X$. This proves that r is bounded. To prove that r is continuous, let $a \in X$. Since $f(j)$ is continuous, there exists a neighborhood U of the point a such that

$$|[f(j)](x) - [f(j)](a)| < \varepsilon$$

holds for all $x \in U$. Then we have

$$\begin{aligned} |r(x) - r(a)| &\leq |[f(j)](x) - [f(j)](a)| + |[f(j)](x) - r(x)| \\ &\quad + |[f(j)](a) - r(a)| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \end{aligned}$$

for all $x \in U$. Since $\varepsilon > 0$ and $a \in X$ are arbitrary, this proves that r is continuous. $\square$

LEMMA 7.8. *A pseudo-metric space (X, d) is totally bounded iff every sequence in X has a Cauchy subsequence.*

Proof: Necessity. Assume that X is totally bounded and consider any given sequence

$$f: N \rightarrow X.$$

For each positive integer n , we shall construct by induction a subsequence

$$f_n: N \rightarrow X$$

of f such that f_n is a subsequence of f_{n-1} with $f_0 = f$, and satisfies the inequality

$$d[f_n(i), f_n(j)] < 1/n$$

for all positive integers i and j .

Since X is totally bounded, it is the union of a finite number of its open balls of radius $\frac{1}{2}$. Hence the sequence f must be frequently in one of these balls. This implies the existence of a subsequence f_1 of f satisfying our condition. Next, let $n > 1$ and assume that the subsequence f_{n-1} of f has been constructed. Since X is the union of a finite number of its open balls of radius $1/2n$, the sequence f_{n-1} must be frequently in one of these

balls. Hence f_{n-1} has a subsequence f_n satisfying the condition. This completes the inductive construction of subsequences $\{f_n \mid n \in N\}$ of f .

Now define a subsequence

$$f_*: N \rightarrow X$$

of f by the classical diagonal method. In other words, f_* is defined by taking

$$f_*(n) = f_n(n)$$

for every $n \in N$. To prove that f_* is a Cauchy sequence in X , let ε denote any given positive real number. Choose an integer $n > 1/\varepsilon$. Then we have

$$d[f_*(i), f_*(j)] < \varepsilon$$

for all integers $i > n$ and $j > n$. Hence f_* is a Cauchy subsequence of f .

Sufficiency. Assume that X is not totally bounded. Then there exists a positive real number r such that no finite subset F of X satisfies

$$X = \bigcup \{N_r(x) \mid x \in F\}.$$

Because of this fact, one can construct inductively a sequence $f: N \rightarrow X$ such that

$$d[f(i), f(j)] \geq r$$

for all positive integers i and j . This sequence f obviously has no Cauchy subsequence. ||

LEMMA 7.9. *If a Cauchy sequence $f: N \rightarrow X$ in a pseudo-metric space (X, d) has a subsequence $g: N \rightarrow X$ which converges to a point $p \in X$, then f converges to the point p .*

Proof. Let ε denote an arbitrary positive real number. Since f is a Cauchy sequence, there exists a positive integer k such that

$$d[f(i), f(j)] < \varepsilon/2$$

holds for all integers $i > k$ and $j > k$. Since g converges to p , there exists a positive integer l such that

$$d[p, g(m)] < \varepsilon/2$$

holds for all integers $m > l$. Since g is a subsequence of f , there exists a cofinal function $h: N \rightarrow N$ with

$$g = f \circ h.$$

Since h is cofinal, there exists an $m \in N$ satisfying $m > l$ and $h(m) > k$. Then we have

$$d[p, f(n)] \leq d[p, g(m)] + d\{f[h(m)], f(n)\} < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for every $n > k$. This proves that f converges to p . $\parallel$

THEOREM 7.10. *A pseudo-metric space is compact iff it is complete and totally bounded.*

Proof: Necessity. Assume that (X, d) is a compact pseudo-metric space. By (5.7), X is totally bounded. To prove the completeness of X , consider any Cauchy sequence $f: N \rightarrow X$. Since X is compact, it follows from (5.11) that f has a subsequence $g: N \rightarrow X$ which converges to a point $p \in X$. According to (7.9), f converges to p . Hence X is complete.

Sufficiency. Assume that (X, d) is a pseudo-metric space which is complete and totally bounded. To prove that X is compact, consider an arbitrary sequence $f: N \rightarrow X$. Since X is totally bounded, it follows from (5.8) that f has a Cauchy subsequence $g: N \rightarrow X$. Since X is complete, g converges to a point $p \in X$. This proves that X is sequentially compact. Hence it follows from (5.11) that X is compact. $\parallel$

A remarkable feature of this theorem (7.10) is that compactness, a topological property, of a pseudo-metric space is equivalent to the conjunction of two metric properties, namely, completeness and total boundedness.

THEOREM 7.11. (BAIRE CATEGORY THEOREM). *The intersection of any countable family of dense open subsets of a complete pseudo-metric space X is dense in X .*

Proof. Consider an arbitrary sequence $\{G_n \mid n \in N\}$ of dense open subsets G_n of a complete pseudo-metric space X and let

$$G = \bigcap \{G_n \mid n \in N\}.$$

To prove that G is dense in X , let U denote any nonempty open subset of X . We have to prove that $U \cap G$ is nonempty.

Since G_1 is dense in X , the open subset $U \cap G_1$ is not empty. Select a point x_1 in $U \cap G_1$. Then there exists a positive real number $\delta_1 < 1$ such that the closure $\text{Cl}[N_{\delta_1}(x_1)]$ of the open ball $N_{\delta_1}(x_1)$ is contained in the open set $U \cap G_1$. Next, since G_2 is dense in X , the open set $N_{\delta_1}(x_1) \cap G_2$. Then there exists a positive real number $\delta_2 < 1/2$ such that the closure $\text{Cl}[N_{\delta_2}(x_2)]$ of the open ball $N_{\delta_2}(x_2)$ is contained in the open set $N_{\delta_1}(x_1) \cap G_2$. Proceeding inductively, we obtain for every positive integer n a point x_n and a positive real number $\delta_n < 1/n$ such that the

closure $\text{Cl}[N_{\delta_n}(x_n)]$ of the open ball $N_{\delta_n}(x_n)$ is contained in the open set $N_{\delta_{n-1}}(x_{n-1}) \cap G_n$. Thus we obtain a sequence

$$f = \{x_1, x_2, \dots, x_n, \dots\}.$$

To prove that f is a Cauchy sequence, let ε denote an arbitrary positive real number. Select a positive integer $k > 2/\varepsilon$. According to the construction of the sequence, we have $x_n \in N_{\delta_k}(x_k)$ for every $n > k$. Hence we obtain

$$d(x_i, x_j) \leq d(x_i, x_k) + d(x_j, x_k) < \delta_k + \delta_k < \frac{2}{k} < \varepsilon$$

for all integers $i > k$ and $j > k$. This proves that f is a Cauchy sequence.

Since the pseudo-metric space X is complete, the Cauchy sequence f converges to a point $p \in X$. It remains to prove $p \in U \cap G$.

For this purpose, let m denote any positive integer. By the construction of the sequence f , we have

$$x_n \in \text{Cl}[N_{\delta_{m+1}}(x_{m+1})] \subset N_{\delta_m}(x_m) \cap G_m \subset U \cap G_m$$

for every integer $n > m$. This implies that

$$p \in \text{Cl}[N_{\delta_{m+1}}(x_{m+1})] \subset U \cap G_m$$

holds for every positive integer m and hence $p \in U \cap G$. $\parallel$

EXERCISES

- 7A. Prove that a pseudo-metric space X is complete iff every infinite totally bounded subset of X has at least one accumulation point in X .
- 7B. Prove that a pseudo-metric space X is complete iff, for every sequence $\{C_n \mid n \in N\}$ of non-empty closed subsets of X satisfying

$$C_{n+1} \subset C_n, \lim_{n \rightarrow \infty} \delta(C_n) = 0,$$

the intersection $\bigcap \{C_n \mid n \in N\}$ is not empty

- 7C. Let X be a topological space and (Y, d) be a metric space. A map $f: X \rightarrow Y$ is said to be *bounded* iff its image $f(X)$ is a bounded subset of Y . Consider the set $\Omega = C(X, Y)$ of all bounded maps from X into Y . Define a function $d^*: \Omega^2 \rightarrow R$ by taking

$$d^*(f, g) = \sup \{d[f(x), g(x)] \mid x \in X\}$$

for all maps f and g in Ω . Prove that d^* is a metric and makes Ω a complete metric space.

7D. Prove that the intersection of any countable family of dense open subsets of a locally compact regular space X is dense in X .

7E. A topological space X is said to be of the *first category* (or *meager*) iff it is the union of a countable family of nowhere dense subsets of X . A topological space is said to be of the *second category* iff it is not of the first category. Prove that every subspace of a meager space is meager and that the union of every countable family of meager subspaces is meager. Verify that the theorem (7.11) can be stated as follows: *Every complete pseudo-metric space is of the second category.*

7F. Let (X, d) denote any complete metric space. Prove that every function $f: X \rightarrow X$ which satisfies

$$d[f(a), f(b)] < kd(a, b)$$

for some real number $k < 1$ and all points a and b of X has a unique *fixed point*, that is, a point $x \in X$ with $f(x) = x$. [K-F, p. 43].

8. COMPLETIONS

By a *pseudo-metric completion* of a pseudo-metric space X , we mean an isometric imbedding

$$i: X \rightarrow Y$$

of X into a complete pseudo-metric space Y such that the image $i(X)$ is dense in Y . Similarly, by a *metric completion* of a metric space X , we mean an isometric imbedding $i: X \rightarrow Y$ of X into a complete metric space Y such that the image $i(X)$ is dense in Y .

Let (X, d) denote an arbitrarily given pseudo-metric space. We will construct a pseudo-metric completion of X . For this purpose, let us consider the set

$$Y = CS(X, d)$$

of all Cauchy sequences in the pseudo-metric space X .

LEMMA 8.1. *For any two Cauchy sequences $f, g: N \rightarrow X$ in a given pseudo-metric space (X, d) , the sequence*

$$\sigma(f, g): N \rightarrow R$$

of real numbers defined by

$$[\sigma(f, g)](n) = d[f(n), g(n)]$$

for all $n \in N$ converges to a real number which will be denoted by $e(f, g)$.

Proof. Let ε denote an arbitrary positive real number. Since f and g are Cauchy sequences, there exists a positive integer k such that

$$d[f(i), f(j)] < \varepsilon/2, \quad d[g(i), g(j)] < \varepsilon/2$$

hold for all integers $i > k$ and $j > k$. Hence we obtain

$$\begin{aligned} |[\sigma(f, g)](i) - [\sigma(f, g)](j)| &\leq |d[f(i), g(i)] - d[f(j), g(j)]| \\ &\leq |d[f(i), g(i)] - d[f(j), g(i)]| + |d[f(j), g(i)] - d[f(j), g(j)]| \\ &\leq d[f(i), f(j)] + d[g(i), g(j)] < \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all integers $i > k$ and $j > k$. This proves that $\sigma(f, g)$ is a Cauchy sequence of real numbers and hence converges to a real number $e(f, g)$. $\parallel$

The assignment $(f, g) \rightarrow e(f, g)$ defines a function

$$e: Y^2 \rightarrow R.$$

LEMMA 8.2. *The function $e: Y^2 \rightarrow R$ is a pseudo-metric in the set $Y = CS(X, d)$.*

Proof. Since (M2) is obviously satisfied, it remains to verify the triangle inequality (M1). For this purpose, let f, g, h denote any three Cauchy sequences in X . Let ε denote an arbitrary positive real number. Since the sequences $\sigma(f, g)$, $\sigma(f, h)$, and $\sigma(g, h)$ converge to $e(f, g)$, $e(f, h)$, and $e(g, h)$ respectively, there exists a positive integer k such that

$$\begin{aligned} e(f, g) &\leq d[f(n), g(n)] + \varepsilon \\ d[f(n), h(n)] &\leq e(f, h) + \varepsilon \\ d[g(n), h(n)] &\leq e(g, h) + \varepsilon \end{aligned}$$

hold for all integers $n > k$. Select any integer $n > k$. Using (M1) for d , we obtain

$$\begin{aligned} e(f, g) &\leq d[f(n), h(n)] + d[g(n), h(n)] + \varepsilon \\ &\leq e(f, h) + e(g, h) + 3\varepsilon. \end{aligned}$$

Since this holds for all positive real numbers ε , we have

$$e(f, g) \leq e(f, h) + e(g, h).$$

Hence e is a pseudo-metric in Y . $\parallel$

This pseudo-metric $e: Y^2 \rightarrow R$ makes the set $Y = CS(X, d)$ a pseudo-metric space which will be called the *space of all Cauchy sequences* in the given pseudo-metric space (X, d) .

If X consists of more than one point, this pseudo-metric space $CS(X, d)$ is not a metric space. In fact, let a and b denote any two distinct points in X . Then $f = (a, a, a, \dots)$ and $g = (b, a, a, \dots)$ are two distinct Cauchy sequences in X with $e(f, g) = 0$.

Next, let us define a function

$$i: X \rightarrow Y = CS(X, d)$$

by taking $i(x)$ to be the Cauchy sequence $(x, x, x, \dots)$ for every point x of X .

LEMMA 8.3. *This function $i: X \rightarrow Y$ is an isometric imbedding which will be called the canonical isometric imbedding of the given pseudo-metric space X into its space $Y = CS(X, d)$ of all Cauchy sequences.*

Proof. It is obvious from the definition that i is injective. By the definition of the pseudo-metric e , we have

$$e[i(a), i(b)] = d(a, b)$$

for any two points a and b of X . Hence i is isometric. Then it follows from (6.4) that i is an isometric imbedding. $\parallel$

LEMMA 8.4. *The image $i(X)$ of the canonical isometric imbedding $i: X \rightarrow Y$ of a pseudo-metric space X into its space Y of all Cauchy sequences is dense in Y .*

Proof. Let $f: N \rightarrow X$ denote any Cauchy sequence in X and ε be an arbitrary positive real number. Since f is a Cauchy sequence, there exists a positive integer k such that

$$d[f(m), f(n)] < \varepsilon/2$$

holds for all integers $m > k$ and $n > k$. Select an arbitrary integer $m > k$ and set $x = f(m) \in X$. Then we obtain

$$d\{[i(x)](n), f(n)\} = d[f(m), f(n)] < \varepsilon/2$$

for every integer $n > k$. This implies

$$e[i(x), f] = \lim_{n \rightarrow \infty} d\{[i(x)](n), f(n)\} \leq \varepsilon/2 < \varepsilon.$$

Since $f \in Y$ and $\varepsilon > 0$ are arbitrary, this proves that $i(X)$ is dense in Y . $\parallel$

LEMMA 8.5. *The space $Y = CS(X, d)$ of all Cauchy sequences in a pseudo-metric space (X, d) is complete.*

Proof. Let $f: N \rightarrow Y$ denote any Cauchy sequence in Y . Since the image $i(X)$ of the canonical isometric imbedding $i: X \rightarrow Y$ is dense in Y , there exists for each positive integer n a point $x_n \in X$ such that

$$e[f(n), i(x_n)] < 1/n$$

holds. Thus we obtain a sequence $g: N \rightarrow X$ defined by $g(n) = x_n$ for every $n \in N$.

To prove that g is a Cauchy sequence in X , let ε denote an arbitrary positive real number. Since f is a Cauchy sequence in Y , there exists a positive integer $k > 3/\varepsilon$ such that

$$e[f(r), f(s)] < \varepsilon/3$$

holds for all integers $r > k$ and $s > k$. Since $i: X \rightarrow Y$ is isometric, we have

$$\begin{aligned} d[g(r), g(s)] &= d(x_r, x_s) = e[i(x_r), i(x_s)] \\ &\leq e[f(r), f(s)] + e[f(r), i(x_r)] + e[f(s), i(x_s)] \\ &< \frac{\varepsilon}{3} + \frac{1}{r} + \frac{1}{s} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for all integers $r > k$ and $s > k$. This proves that g is a Cauchy sequence in X . Hence we have $g \in Y$.

It remains to prove that the given Cauchy sequence $f: N \rightarrow Y$ converges to the point $g \in Y$. For this purpose, let ε denote any positive real number. Since $g: N \rightarrow X$ is a Cauchy sequence, there exists a positive integer $k > 2/\varepsilon$ such that

$$d(x_m, x_n) < \varepsilon/2$$

holds for all integers $m > k$ and $n > k$. This implies that

$$e[g, i(x_n)] = \lim_{m \rightarrow \infty} d(x_m, x_n) \leq \varepsilon/2$$

holds for every integer $n > k$. Hence we obtain

$$\begin{aligned} e[f(n), g] &\leq e[f(n), i(x_n)] + e[g, i(x_n)] \\ &< \frac{1}{n} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for every integer $n > k$. This proves that the sequence f converges to the point g . Hence Y is complete. $\square$

Summarizing the lemmas, we obtain the following theorem.

THEOREM 8.6. *The canonical isometric imbedding*

$$i: X \rightarrow CS(X, d)$$

of a pseudo-metric space (X, d) into its space $CS(X, d)$ of all Cauchy sequences is a pseudo-metric completion of the pseudo-metric space X .

Before taking up the construction of metric completions for metric spaces, let us first establish the following lemma.

LEMMA 8.7. *Any isometric image of a complete pseudo-metric space is complete.*

Proof. Let $j: X \rightarrow Y$ denote any surjective isometric map from a complete pseudo-metric space (X, d) onto a pseudo-metric space (Y, e) . We have to prove that Y is complete.

For this purpose, consider an arbitrary Cauchy sequence $g: N \rightarrow Y$ in Y . Since j is surjective, there exists a sequence $f: N \rightarrow X$ satisfying $g = j \circ f$. Since j is isometric, we have

$$e[g(m), g(n)] = e\{j[f(m)], j[f(n)]\} = d[f(m), f(n)]$$

for all integers $m \in N$ and $n \in N$. Since g is a Cauchy sequence, this implies that f is a Cauchy sequence in X .

Since X is complete, the Cauchy sequence f converges to a point $p \in X$. Let $q = j(p) \in Y$. Since j is isometric, we have

$$e[g(n), q] = e\{j[f(n)], j(p)\} = d[f(n), p]$$

for every positive integer n . Since f converges to p , this implies that q converges to q . Hence Y is complete. $\parallel$

Now let (X, d) be an arbitrarily given metric space and let

$$(Y, e) = CS(X, d)$$

denote the pseudo-metric space of all Cauchy sequences in X . Consider the reduced metric space (Y^*, e^*) of (Y, e) as defined in the example (1) of § 6.

THEOREM 8.8. *For an arbitrary metric space (X, d) , the composition*

$$j = p \circ i: X \rightarrow Y^*$$

of the canonical isometric imbedding $i: X \rightarrow Y$ of X into its space (Y, e) of all Cauchy sequences and the natural projection $p: Y \rightarrow Y^$ of Y onto its reduced metric space (Y^*, e^*) is a metric completion of X .*

Proof. Since both i and p are isometric, so is their composition j . Since X is a metric space, it follows from (6.5) that j is an imbedding of X into Y^* . Since $i(X)$ is dense in Y and p is surjective, it follows that $j(X)$ is dense in Y^* . Finally, as an isometric image of a complete space Y , Y^* is complete. Hence, j is a metric completion of X . $\parallel$

As another proof of the existence of metric completions, we have the following theorem.

THEOREM 8.9. *For an arbitrary metric space (X, d) , the canonical imbedding $q: X \rightarrow C(X)$ of X into the metric space $C(X)$ of all bounded continuous real-valued functions on X with a base point $z \in X$ defines a metric completion*

$$h: X \rightarrow Y = \text{Cl}[q(X)] \subset C(X)$$

of the metric space X .

Proof. In the example (2) of § 6, we established the fact that q is an isometric imbedding of X into $C(X)$. Since h is defined by q and Y is the closure of $q(X)$ in $C(X)$, it follows that h is an isometric imbedding and that $h(X)$ is dense in Y . As a closed subspace of a complete metric space $C(Y)$, Y is a complete metric space. Hence h is a metric completion of X . ||

Having established the existence of pseudo-metric and metric completions, we now turn to the problem whether completions are essentially unique. For pseudo-metric completions, the answer is obviously negative. In fact, in the two pseudo-metric completions $i: X \rightarrow Y$ and $j: X \rightarrow Y^*$ of any given metric space X in (8.8), the spaces Y and Y^* are even not homeomorphic in case X consists of more than one point.

Before taking up the uniqueness of metric completions of metric spaces, let us first study two special properties of complete metric spaces. In the proofs of these, the property of being a Fréchet space has to be used.

THEOREM 8.10. *Every uniformly continuous function $f: A \rightarrow Y$ from any dense subspace A of a pseudo-metric space (X, d) into a complete metric space (Y, e) has a uniquely determined continuous extension*

$$f^*: X \rightarrow Y$$

over X . Furthermore, f^ is uniformly continuous.*

Proof. Let us define a function $f^*: X \rightarrow Y$ as follows. Let x denote an arbitrary point of X . Since A is dense in X , there exists a sequence $\phi: N \rightarrow A$ which converges to the point x . Then the composition

$$\phi^* = f \circ \phi: N \rightarrow Y$$

is a sequence in the space Y .

To prove that ϕ^* is a Cauchy sequence, let ε denote an arbitrary positive real number. Since f is uniformly continuous, there exists a positive real number δ such that

$$e[f(u), f(v)] < \varepsilon$$

holds for any two points u and v of A satisfying $d(u, v) < \delta$. Since ϕ converges to x , it is a Cauchy sequence according to (7.1). Hence there exists a positive integer k such that

$$d[\phi(i), \phi(j)] < \delta$$

holds for all integers $i > k$ and $j > k$. This implies that

$$e[\phi^*(i), \phi^*(j)] = e\{f[\phi(i)], f[\phi(j)]\} < \varepsilon$$

holds for all integers $i > k$ and $j > k$. Hence ϕ^* is a Cauchy sequence. Since Y is a complete metric space, this Cauchy sequence ϕ^* converges to a unique point y_ϕ in Y .

Next consider any other sequence $\psi: N \rightarrow A$ which converges to the same point x as does ϕ . Then we obtain a Cauchy sequence $\psi^* = f \circ \psi: N \rightarrow Y$ which converges to a unique point y_ψ in Y . To prove $y_\phi = y_\psi$, let ε denote any positive real number. Since ϕ^* converges to y_ϕ and ψ^* converges to y_ψ , there exists a positive integer k such that

$$e[y_\phi, \phi^*(n)] < \varepsilon/3, \quad e[y_\psi, \psi^*(n)] < \varepsilon/3$$

hold for all integers $n > k$. Since f is uniformly continuous, there exists a positive real number δ such that

$$e[f(u), f(v)] < \varepsilon/3$$

holds for any two points u and v of X satisfying $d(u, v) < \delta$. Since ϕ and ψ both converge to the point x , there exists an integer $n > k$ such that

$$d[x, \phi(n)] < \delta/2, \quad d[x, \psi(n)] < \delta/2$$

are satisfied. Hence we obtain

$$d[\phi(n), \psi(n)] \leq d[x, \phi(n)] + d[x, \psi(n)] < \delta.$$

Consequently, we have

$$\begin{aligned} e(y_\phi, y_\psi) &\leq e[y_\phi, \phi^*(n)] + e[y_\psi, \psi^*(n)] + e[\phi^*(n), \psi^*(n)] \\ &= e[y_\phi, \phi^*(n)] + e[y_\psi, \psi^*(n)] + e\{f[\phi(n)], f[\psi(n)]\} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since this is true for all positive real numbers ε , this implies $e(y_\phi, y_\psi) = 0$. Since Y is a metric space, this proves $y_\phi = y_\psi$.

Therefore, the point y_ϕ does not depend on the choice of the sequence ϕ and may be denoted by $f^*(x)$. This assignment $x \rightarrow f^*(x)$ defines a function

$$f^*: X \rightarrow Y.$$

To prove the uniform continuity of f^* , let ε denote an arbitrary positive real number. Since $f: A \rightarrow Y$ is uniformly continuous, there exists a positive real number δ such that

$$e[f(u), f(v)] < \varepsilon/3$$

holds for any two points u and v satisfying $d(u, v) < 3\delta$. Now let p and q denote any two points of X satisfying $d(p, q) < \delta$. Select sequences $\phi, \psi: N \rightarrow A$ converging to p and q respectively. Then the sequences $\phi^* = f \circ \phi$ and $\psi^* = f \circ \psi$ converge to $f^*(p)$ and $f^*(q)$ respectively. Hence there exists a positive integer n such that the following four inequalities

$$\begin{aligned} d[p, \phi(n)] &< \delta, & d[q, \psi(n)] &< \delta \\ e[f^*(p), \phi^*(n)] &< \varepsilon/3, & e[f^*(q), \psi^*(n)] &< \varepsilon/3 \end{aligned}$$

are satisfied. Then we have

$$d[\phi(n), \psi(n)] \leq d[p, \phi(n)] + d[q, \psi(n)] + d(p, q) < 3\delta$$

and hence we obtain

$$\begin{aligned} e[f^*(p), f^*(q)] &\leq e[f^*(p), \phi^*(n)] + e[f^*(q), \psi^*(n)] + e[\phi^*(n), \psi^*(n)] \\ &= e[f^*(p), \phi^*(n)] + e[f^*(q), \psi^*(n)] + e\{f[\phi(n)], f[\psi(n)]\} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This proves that f^* is uniformly continuous.

To prove that f^* is an extension of f , let x denote an arbitrary point of A . Then the constant sequence $\phi: N \rightarrow A$ defined by $\phi(n) = x$ for all $n \in N$ converges to x and the sequence $\phi^* = f \circ \phi$ converges to $f^*(x)$. Since ϕ^* is the constant sequence defined by $\phi^*(n) = f(x)$ for all $n \in N$, ϕ^* also converges to $f(x)$. Since Y is a metric space, this implies that $f^*(x) = f(x)$. This proves that f^* is an extension of f .

To prove the uniqueness of f^* , let $g: X \rightarrow Y$ denote any continuous extension of f and consider an arbitrary point of X . Select a sequence $\phi: N \rightarrow A$ which converges to x . By the definition of f^* , the sequence $\phi^* = f \circ \phi$ converges to $f^*(x)$. On the other hand, since g is a continuous extension of f , the sequence

$$g \circ \phi = f \circ \phi = \phi^*$$

converges to $g(x)$. Since Y is a metric space, this implies $g(x) = f^*(x)$. Since x is any point of X , we have $g = f^*$. This proves the uniqueness of f^* . ||

COROLLARY 8.11. *If the function $f: A \rightarrow Y$ in (8.10) is isometric, then so is its continuous extension $f^*: X \rightarrow Y$.*

Proof. Assume that f is isometric and consider any two points p and q of X . Select sequences $\phi, \psi: N \rightarrow A$ converging to p and q respectively. Then the sequences $\phi^* = f \circ \phi$ and $\psi^* = f \circ \psi$ converge to $f^*(p)$ and $f^*(q)$ respectively. Since pseudo-metrics are continuous according to (2.2), we have

$$\begin{aligned} e[f^*(p), f^*(q)] &= \lim_{n \rightarrow \infty} e[\phi^*(n), \psi^*(n)] \\ &= \lim_{n \rightarrow \infty} e\{f[\phi(n)], f[\psi(n)]\} \\ &= \lim_{n \rightarrow \infty} d[\phi(n), \psi(n)] = d(p, q). \end{aligned}$$

This proves that f^* is isometric. $\parallel$

THEOREM 8.12. *Every complete subspace of a metric space is closed.*

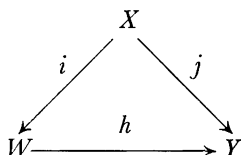
Proof. Let A denote any complete subspace of a metric space X . To prove that A is closed in X , let x denote any point of the closure $\text{Cl}(A)$ of A in X . Then there exists a sequence $\phi: N \rightarrow A$ which converges to the point x . By (7.1), ϕ is a Cauchy sequence. Since A is complete, ϕ converges to a point $a \in A$. Since X is a metric space, we must have $x = a \in A$. This proves that A is closed in X . $\parallel$

For the uniqueness of metric completion, we have the following theorem.

THEOREM 8.13. *Let $j: X \rightarrow Y$ denote any metric completion of a metric space X . Then, for every pseudo-metric completion $i: X \rightarrow W$ of X , there exists a uniquely determined surjective isometric map $h: W \rightarrow Y$ such that*

$$h \circ i = j$$

holds in the following triangle:



Proof. Define a function $f: i(X) \rightarrow Y$ by taking

$$f(w) = j[i^{-1}(w)]$$

for every $w \in i(X)$. Since i and j are isometric, so is f . By (8.10), f has a uniquely determined continuous extension $h: W \rightarrow Y$. By (8.11), h is

isometric. By (8.7), $h(W)$ is complete. Then it follows from (8.12) that $h(W)$ is closed in Y . Since the closed set $h(W)$ contains the set

$$h[i(X)] = f[i(X)] = j(X)$$

which is dense in Y , we must have $h(W) = Y$. Hence, h is surjective. Since h is an extension of f , we have $h \circ i = j$.

It remains to prove the uniqueness of h . For this purpose, let $g: W \rightarrow Y$ denote any map satisfying $g \circ i = j$. Then g is a continuous extension of f . By (8.10), we have $g = h$. $\parallel$

If W is a metric space, then it follows from (6.5) that $h: W \rightarrow Y$ is an isometric homeomorphism. Hence we have the following corollary.

COROLLARY 8.14. *For any two metric completions $i: X \rightarrow W$ and $j: X \rightarrow Y$ of a metric space X , there exists a uniquely determined isometric homeomorphism $h: W \rightarrow Y$ satisfying $h \circ i = j$.*

Thus metric completions are essentially unique.

EXERCISES

- 8A. Let X denote any complete metric space. Verify that the identity map $i: X \rightarrow X$ is a metric completion of X and that every metric completion $j: X \rightarrow Y$ is an isometric homeomorphism.
- 8B. For any pseudo-metric completion $i: X \rightarrow Y$ of a pseudo-metric space X , prove that Y is separable iff X is so.
- 8C. A metric space X is said to be *absolutely closed* iff every isometric image of X into any metric space Y is closed in Y . Prove that a metric space is complete iff it is absolutely closed.
- 8D. A metrizable space X is said to be *topologically complete* iff its topology can be defined by a metric $d: X^2 \rightarrow R$ such that the metric space (X, d) is complete. A metrizable space X is said to be an *absolute G_δ* iff it is a G_δ in every metric space in which it is topologically imbedded. Prove that a metrizable space is topologically complete iff it is an absolute G_δ .

Chapter V: UNIFORMITY AND BOUNDEDNESS

In the preceding chapter we studied several nontopological properties of metric spaces and their maps such as boundedness, total boundedness, completeness, and uniform continuity. Unfortunately, these interesting properties are lost in the generalization from metric spaces to topological spaces. Now we are concerned with other generalizations of metric spaces in which these properties are still meaningful. There are two different kinds of properties defined by pseudo-metrics, namely, the “lower-end properties” and the “upper-end properties.” The former are those expressed by means of all arbitrarily small positive real numbers while the latter are those defined by some sufficiently large positive real numbers. To generalize the lower-end properties, we have Weil’s theory uniform spaces [We]; while to generalize the upper-end properties, we will give an axiomatic approach to boundedness introduced by the present author about twenty years ago, [Hu 2].

1. UNIFORMITIES

For an arbitrarily given set X , let us consider the Cartesian square

$$X^2 = X \times X$$

of X . We recall that the *diagonal* of X^2 is the subset

$$\Delta = \Delta(X) = \{(x, x) \in X^2 \mid x \in X\}$$

of X^2 . The subsets of X^2 are called the *relations* in X . For each relation U in X , the *inverse* U^{-1} of U is defined by

$$U^{-1} = \{(x, y) \in X^2 \mid (y, x) \in U\}.$$

For any two relations U and V in X , the *composition* $U \circ V$ of U and V is defined by

$$U \circ V = \{(x, y) \in X^2 \mid (x, w) \in U \text{ and } (w, y) \in V \text{ for some } w \in X\}.$$

By a *uniformity* in a given set X , we mean a nonempty family $\mathcal{U}$ of relations in X , that is, subsets of X^2 , satisfying the following five conditions:

- (U1) Every $U \in \mathcal{U}$ contains the diagonal Δ of X^2 .
- (U2) For every $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ with $V \circ V \subset U$.
- (U3) $U \in \mathcal{U}$ and $V \in \mathcal{U}$ imply $U \cap V \in \mathcal{U}$.
- (U4) Every subset V of X^2 which contains a member $U \in \mathcal{U}$ is in $\mathcal{U}$.
- (U5) For every $U \in \mathcal{U}$, we have $U^{-1} \in \mathcal{U}$.

EXAMPLES OF UNIFORMITIES:

(1) *The uniformity of a pseudo-metric space.* Consider an arbitrary pseudo-metric space (X, d) . Define a family $\mathcal{U}$ of subsets of X^2 as follows: a subset U of X^2 is in $\mathcal{U}$ iff there exists a positive real number r such that U contains every point $(a, b) \in X^2$ satisfying $d(a, b) < r$. To check the conditions (U1)–(U5), we note that (M2) implies (U1) and (M1) implies (U2). The conditions (U3) and (U4) are clearly satisfied. The condition (U5) is a consequence of (M4). Hence the family $\mathcal{U}$ is a uniformity in the set X , called the *uniformity defined by the pseudo-metric d* or the *uniformity of the pseudo-metric space (X, d)* .

(2) *The uniformity of a topological group.* By a *topological group*, we mean a group X as defined in [H3, Chapter III] together with a topology τ in X such that the function $g: X^2 \rightarrow X$ defined by

$$g(a, b) = ab^{-1}$$

for all points (a, b) of X^2 is continuous with respect to the topology τ in X and the product topology in X^2 . Define a family $\mathcal{U}$ of subsets of X^2 for an arbitrarily given topological group X as follows: a subset U of X^2 is in $\mathcal{U}$ iff there exists a neighborhood M of the *unit*, or *neutral element*, e of X such that U contains every point $(a, b) \in X^2$ satisfying $ab^{-1} \in M$. To check the conditions (U1)–(U5), we first note that (U1) holds because of $xx^{-1} = e$ for every $x \in X$. The conditions (U3) and (U4) are clearly satisfied. The conditions (U5) and (U2) are consequences of the continuity of the functions $i: X \rightarrow X$ and $j: X^2 \rightarrow X$ defined by

$$i(a) = a^{-1}, \quad j(a, b) = ab$$

for all points a and b of X . Hence $\mathcal{U}$ is a uniformity in X , called the *uniformity of the topological group X* .

By a *basis* of a uniformity $\mathcal{U}$ in a set X , we mean a subfamily $\mathcal{B}$ of $\mathcal{U}$ such that every member of $\mathcal{U}$ contains a member of $\mathcal{B}$. Because of the condition (U4), every basis $\mathcal{B}$ of $\mathcal{U}$ completely determines $\mathcal{U}$ as follows: a subset U of X^2 is in $\mathcal{U}$ iff it contains a member of $\mathcal{B}$.

By a *sub-basis* of a uniformity $\mathcal{U}$ in a set X , we mean a subfamily $\mathcal{S}$ of $\mathcal{U}$ such that the intersection of the members of the finite subfamilies of

$\mathcal{S}$ form a basis of $\mathcal{U}$. Because of the conditions (U3) and (U4), every sub-basis $\mathcal{S}$ of $\mathcal{U}$ completely determines $\mathcal{U}$ as follows: a subset U of X^2 is in $\mathcal{U}$ iff there exists a finite subfamily of $\mathcal{S}$, say $\mathcal{F} = \{V_1, \dots, V_n\}$, satisfying

$$V_1 \cap \dots \cap V_n \subset U.$$

EXAMPLES OF BASES AND SUB-BASES:

(a) Consider the uniformity $\mathcal{U}$ of a pseudo-metric space (X, d) . Then the subfamily $\mathcal{B} = \{U_n \mid n \in N\}$ of $\mathcal{U}$ defined by

$$U_n = \{(a, b) \in X \mid d(a, b) < 1/n\}$$

for every $n \in N$ is obviously a basis of $\mathcal{U}$.

(b) Consider the uniformity $\mathcal{U}$ of a topological group X . Let σ denote an arbitrary local sub-basis at the unit $e \in X$ of the topology τ in X . Then the subfamily

$$\mathcal{S} = \{U_M \mid M \in \sigma\}$$

of $\mathcal{U}$ defined by

$$U_M = \{(a, b) \in X^2 \mid ab^{-1} \in M\}$$

for every $M \in \sigma$ is obviously a sub-basis of $\mathcal{U}$.

Now let $\mathcal{U}$ denote an arbitrary uniformity in any given set X . For each subset A of X and every $U \in \mathcal{U}$, let $U(A)$ denote the subset of X which consists of all points $x \in X$ such that $(a, x) \in U$ holds for some point $a \in A$. In particular, if A consists of a single point $a \in X$, then we have

$$U(a) = U(\{a\}) = \{x \in X \mid (a, x) \in U\}.$$

Define a family τ of subsets of the given set X as follows: a subset G of X is in τ iff, for every point $a \in G$, there exists a member U of the uniformity $\mathcal{U}$ satisfying $U(a) \subset G$. To check the axioms (OS1)–(OS4) for open set, we note that (OS4) is a consequence of (U3) while the other axioms are clearly satisfied. Hence τ is a topology in X called the *topology defined by the uniformity $\mathcal{U}$* .

One can easily verify that the uniformity of a pseudo-metric space X defines the topology of X and that the uniformity of a topological group X defines the topology of X .

By a *uniform space*, we mean a set X furnished with a uniformity $\mathcal{U}$ in X and the topology τ in X defined by the uniformity $\mathcal{U}$.

For examples, pseudo-metric spaces and topological groups are two outstanding classes of uniform spaces.

PROPOSITION 1.1. *A uniform space $(X, \mathcal{U})$ is a Fréchet space iff the intersection of all members of $\mathcal{U}$ is the diagonal Δ of X^2 .*

Proof. Necessity. Assume that the uniform space $(X, \mathcal{U})$ is a Fréchet space and let (a, b) denote an arbitrary point in $X^2 \setminus \Delta$. Then we have $a \neq b$. Since X is a Fréchet space, there exists an open set G of X which contains a but not b . Since the topology of X is defined by the uniformity $\mathcal{U}$, there exists a $U \in \mathcal{U}$ satisfying $U(a) \subset G$. This implies $b \notin U(a)$ and hence $(a, b) \notin U$. Since (a, b) is an arbitrary point of $X^2 \setminus \Delta$, this and (U1) imply that the intersection of all members of $\mathcal{U}$ is Δ .

Sufficiency. Assume that Δ is the intersection of all members of $\mathcal{U}$. To prove that X is a Fréchet space, let a denote any point of X . We will prove that $X \setminus \{a\}$ is an open set of X . For this purpose, let x be an arbitrary point of $X \setminus \{a\}$. Then the point (x, a) of X^2 is not in the diagonal Δ . Hence there exists a $U \in \mathcal{U}$ which does not contain (x, a) . This implies that a is not in $U(x)$. Thus we have $U(x) \subset X \setminus \{a\}$. This proves that $X \setminus \{a\}$ is an open set of X and hence X is a Fréchet space. ||

A member U of a uniformity $\mathcal{U}$ in a set X is said to be *symmetric* iff

$$U^{-1} = U.$$

PROPOSITION 1.2. *For an arbitrary member U of a uniformity $\mathcal{U}$ in a set X , there exists a symmetric member V of $\mathcal{U}$ satisfying*

$$V \subset V \circ V \subset U.$$

Proof. According to the condition (U2) there exists a member W of $\mathcal{U}$ satisfying $W \circ W \subset U$. By the condition (U5), W^{-1} is a member of $\mathcal{U}$. Then it follows from the condition (U3) that the subset

$$V = W \cap W^{-1}$$

is a member of $\mathcal{U}$. Hence V is a symmetric member of $\mathcal{U}$ satisfying

$$V \circ V \subset W \circ W \subset U.$$

Since V contains the diagonal Δ of X^2 by (U1), we also have

$$V = V \circ \Delta \subset V \circ V.$$

This completes the proof. ||

EXERCISES

1A. For arbitrary subsets U, V, W of X^2 , prove the following three assertions:

(a) $(U \circ V)^{-1} = V^{-1} \circ U^{-1}$.

(b) $(U \circ V) \circ W = U \circ (V \circ W)$.

(c) If U is *symmetric*, that is, $U^{-1} = U$, then we have

$$U \circ V \circ U = \bigcup \{U(x) \times U(y) \mid (x, y) \in V\}.$$

1B. For an arbitrary uniform space $(X, \mathcal{U})$ and its topological square X^2 , prove the following two assertions:

(a) The closure of any subset A of X is given by

$$\text{Cl}(A) = \bigcap \{U(A) \mid U \in \mathcal{U}\}.$$

(b) The closure of any subset E of X^2 is given by

$$\text{Cl}(E) = \bigcap \{U \circ E \circ U \mid U \in \mathcal{U}\}.$$

1C. Prove that, for any uniform space $(X, \mathcal{U})$, the interior (with respect to the product topology of X^2) of any member of $\mathcal{U}$ is also a member of $\mathcal{U}$.

1D. For an arbitrary uniform space $(X, \mathcal{U})$, prove the following two assertions:

(a) The subfamily of all open symmetric members of $\mathcal{U}$ is a basis of $\mathcal{U}$.

(b) The subfamily of all closed symmetric members of $\mathcal{U}$ is a basis of $\mathcal{U}$.

1E. Prove that a family $\mathcal{B}$ of subsets of X^2 is a basis of some uniformity in X iff the following four conditions are satisfied.

(a) Every member of $\mathcal{B}$ contains Δ .

(b) For each $U \in \mathcal{B}$, there exists a $V \in \mathcal{B}$ with $V \circ V \subset U$.

(c) For each $U \in \mathcal{B}$, there exists a $V \in \mathcal{B}$ with $V \subset U^{-1}$.

(d) The intersection of any two members of $\mathcal{B}$ contains a member of $\mathcal{B}$.

1F. Prove that a family $\mathcal{S}$ of subsets of X^2 is a sub-basis of some uniformity in X iff the following three conditions are satisfied:

(a) Every member of $\mathcal{S}$ contains Δ .

(b) For each $U \in \mathcal{S}$, there exists a $V \in \mathcal{S}$ with $V \circ V \subset U$.

(c) For each $U \in \mathcal{S}$, there exists a $V \in \mathcal{S}$ with $V \subset U^{-1}$.

1G. For a topological group X , the uniformity $\mathcal{U}$ defined in the example (2) is usually called the *right uniformity* $\mathcal{U}$ of X . There is a *left*

uniformity $\mathcal{V}$ of X defined in the obvious way. Furthermore, $\mathcal{U} \cup \mathcal{V}$ is a sub-basis of a uniformity $\mathcal{W}$ in X called the *two-sided uniformity* of X . Prove that each of the three uniformities $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ defines the topology τ of X .

2. METRIZATION OF UNIFORM SPACES

A uniform space X is said to be *pseudo-metrizable* iff there exists a pseudo-metric

$$d: X^2 \rightarrow R$$

which defines the uniformity $\mathcal{U}$ as well as the topology τ of X . If d is a metric, then we say that the given uniform space X is *metrizable*.

THEOREM 2.1. A UNIFORM SPACE $(X, \mathcal{U})$ is *pseudo-metrizable* iff its uniformity $\mathcal{U}$ has a countable basis.

Proof. Necessity. Assume that the uniform space $(X, \mathcal{U})$ is pseudo-metrizable. Then there exists a pseudo-metric d in X which defines the uniformity $\mathcal{U}$ as well as the topology τ of X . According to the example (a) in § 1, the subfamily $\mathcal{B} = \{U_n \mid n \in N\}$ of $\mathcal{U}$ with

$$U_n = \{(a, b) \in X^2 \mid d(a, b) < 1/n\}$$

is a countable basis of $\mathcal{U}$.

Sufficiency. Assume that the uniformity $\mathcal{U}$ has a countable basis

$$\mathcal{B} = \{U_n \mid n \in N\}.$$

We will construct by induction a sequence

$$\mathcal{C} = \{V_n \mid n \in N\}$$

of symmetric members V_n of $\mathcal{U}$ such that

$$V_n \circ V_n \circ V_n \subset U_n \cap V_{n-1}$$

holds for every $n \in N$ with $V_0 = X^2$.

For this purpose, let n denote any positive integer and assume that V_{n-1} has been constructed. By (U3), $U_n \cap V_{n-1}$ is a member of $\mathcal{U}$. According to (1.2), there exist symmetric members W_n and V_n of $\mathcal{U}$ satisfying

$$W_n \circ W_n \subset U_n \cap V_{n-1}, \quad V_n \circ V_n \subset W_n.$$

Since $\Delta \subset V_n$ holds according to (U1), we have

$$\begin{aligned} V_n \circ V_n \circ V_n &= (V_n \circ V_n) \circ (V_n \circ \Delta) \\ &\subset (V_n \circ V_n) \circ (V_n \circ V_n) \\ &\subset W_n \circ W_n \subset U_n \cap V_{n-1}. \end{aligned}$$

This completes the inductive construction of the sequence $\mathcal{C}$. Since

$$V_n = \Delta \circ V_n \circ \Delta \subset V_n \circ V_n \circ V_n \subset U_n$$

holds for all $n \in N$, $\mathcal{C}$ is a basis of $\mathcal{U}$.

Next, define a function $f: X^2 \rightarrow R$ by taking

$$f(a, b) = \inf \{2^{-n} \mid (a, b) \in V_n\}$$

for every point $(a, b) \in X^2$. Since every V_n is symmetric, so is f , that is,

$$f(a, b) = f(b, a)$$

holds for all points a and b of X .

Next, define a function $d: X^2 \rightarrow R$ by taking

$$d(a, b) = \inf \sum_{i=1}^m f(x_{i-1}, x_i)$$

where the infimum is taken over all possible finite sequences $(x_0, x_1, \dots, x_m)$ of points x_i in X satisfying $x_0 = a$ and $x_m = b$. Since the function f is symmetric, so is the function d .

To verify the condition (M1) for d , let a, b, c denote arbitrary points of X and consider any positive real number ε . Then it follows from the definition of d that there exist two finite sequences $(y_0, y_1, \dots, y_p)$ and $(z_0, z_1, \dots, z_q)$ of points in X satisfying $y_0 = a, y_p = c, z_0 = c, z_q = b$, and

$$\sum_{i=1}^p f(y_{i-1}, y_i) < d(a, c) + \frac{\varepsilon}{2},$$

$$\sum_{j=1}^q f(z_{j-1}, z_j) < d(b, c) + \frac{\varepsilon}{2}.$$

Since the finite sequence $(y_0, y_1, \dots, y_p, z_1, \dots, z_q)$ begins with $y_0 = a$ and ends with $z_q = b$, we have

$$d(a, b) \leq \sum_{i=1}^p f(y_{i-1}, y_i) + \sum_{j=1}^q f(z_{j-1}, z_j) < d(a, c) + d(b, c) + \varepsilon.$$

Since this is true for all positive real numbers ε , we obtain

$$d(a, b) \leq d(a, c) + d(b, c).$$

To verify the condition (M2) for d , consider an arbitrary point x of X . Since $(x, x) \in \Delta \subset V_n$ holds for all $n \in N$, we have

$$f(x, x) = \inf \{2^{-n} \mid n \in N\} = 0.$$

Since (x, x) is a finite sequence which begins with x and ends with x , we have

$$d(x, x) \leq f(x, x) = 0.$$

On the other hand, the function f is clearly nonnegative and hence so is d . This implies $d(x, x) = 0$.

Thus we have established that $d: X^2 \rightarrow R$ is a pseudo-metric in X . It remains to prove that d defines the uniformity $\mathcal{U}$ of the uniform space X . According to the definition given in the example (1) of § 1, it suffices to prove that an arbitrary subset U of X^2 is a member of $\mathcal{U}$ iff it contains

$$P_r = d^{-1}(r) = \{(a, b) \in X^2 \mid d(a, b) < r\}$$

for some positive real number r .

First, let U denote any subset of X^2 which contains P_r for some positive real number r . Choose a positive integer n satisfying $2^{-n} < r$. Let (a, b) denote any point in V_n . Then we have

$$d(a, b) \leq f(a, b) \leq 2^{-n} < r.$$

This implies $V_n \subset P_r \subset U$. By (U4), U is a member of $\mathcal{U}$.

To prove the converse, let us first establish Frink's inequality

$$f(x_0, x_m) \leq 2 \sum_{i=1}^m f(x_{i-1}, x_i)$$

for all finite sequences $(x_0, x_1, \dots, x_m)$ of points of X , [Frink 1]. We will establish this inequality by induction on the positive integer m .

For $m = 1$, Frink's inequality is obvious since f is nonnegative. Let $m > 1$ and assume that Frink's inequality holds for all positive integers less than m . Denote

$$r = \sum_{i=1}^m f(x_{i-1}, x_i).$$

Let j denote the largest nonnegative integer satisfying

$$\sum_{i=1}^j f(x_{i-1}, x_i) \leq r/2$$

and hence

$$\sum_{i=1}^{j+1} f(x_{i-1}, x_i) > r/2.$$

The latter implies

$$\sum_{i=j+2}^m f(x_{i-1}, x_i) < r/2.$$

Because of $0 \leq j < m$, it follows from our inductive hypothesis that we have

$$f(x_0, x_j) \leq 2 \sum_{i=1}^j f(x_{i-1}, x_i) \leq 2(r/2) = r$$

and

$$f(x_{j+1}, x_m) \leq 2 \sum_{i=j+2}^m f(x_{i-1}, x_i) < 2(r/2) = r.$$

Since the function f is nonnegative, we have

$$f(x_j, x_{j+1}) \leq r.$$

Now let n denote the smallest positive integer satisfying $2^{-n} \leq 2r$. Then it follows that the three points (x_0, x_j) , (x_j, x_{j+1}) , and (x_{j+1}, x_m) of X^2 are contained in V_{n+1} . Because of the inclusion

$$V_{n+1} \circ V_{n+1} \circ V_{n+1} \subset V_n,$$

this implies that the point (x_0, x_m) is in V_n . Hence we obtain

$$f(x_0, x_m) \leq 2^{-n} \leq 2r \leq 2 \sum_{i=1}^m f(x_{i-1}, x_i).$$

This completes the inductive proof of Frink's inequality.

Now let U denote an arbitrary member of the uniformity $\mathcal{U}$. Since the sequence $\mathcal{C}$ is a basis of $\mathcal{U}$, there exists a positive integer n such that V_n is contained in U . Let $r = 2^{-n-1}$. We will prove $P_r \subset U$. For this purpose, let (a, b) denote any point in P_r . Then we have $d(a, b) < r$. By the definition of r , there exists a finite sequence $(x_0, x_1, \dots, x_m)$ of points in X satisfying $x_0 = a$, $x_m = b$, and

$$\sum_{i=1}^m f(x_{i-1}, x_i) < r.$$

By Frink's inequality, this implies

$$f(a, b) \leq 2 \sum_{i=1}^m f(x_{i-1}, x_i) < 2r = 2^{-n}.$$

By the definition of the function f , this implies $(a, b) \in V_n$. Hence we have

$$P_r \subset V_n \subset U.$$

This completes the proof that d defines $\mathcal{U}$. $\parallel$

COROLLARY 2.2. *A topological group is pseudo-metrizable iff it satisfies the first axiom of countability.*

The following corollary is a direct consequence of (1.1), (2.1), and (IV, 2.1).

COROLLARY 2.3. *A uniform space $(X, \mathcal{U})$ is metrizable iff its uniformity $\mathcal{U}$ has a countable basis $\mathcal{B}$ such that the intersection of all members of $\mathcal{B}$ is the diagonal Δ of X^2 .*

EXERCISES

- 2A. Prove that a uniformity $\mathcal{U}$ in a set X has a countable basis iff it has a countable sub-basis. Hence a uniform space $(X, \mathcal{U})$ is pseudo-metrizable iff $\mathcal{U}$ has a countable sub-basis.
- 2B. Let $\mathcal{F} = \{(X_\mu, \mathcal{U}_\mu) \mid \mu \in M\}$ denote an arbitrary family of uniform spaces. Consider the Cartesian product $X = \prod_\mu X_\mu$. For each $\mu \in M$ and each $U_\mu \in \mathcal{U}_\mu$, define a subset $W(U_\mu)$ of X by taking

$$W(U_\mu) = \{(f, g) \in X^2 \mid (f(\mu), g(\mu)) \in U_\mu\}.$$

Verify that the family $\mathcal{S} = \{W(U_\mu) \mid \mu \in M, U_\mu \in \mathcal{U}_\mu\}$ is a sub-basis of a uniformity $\mathcal{U}$ in X called the *product* of the uniformities $\mathcal{U}_\mu$. The uniform space $(X, \mathcal{U})$ is called the *uniform product* of the uniform spaces in $\mathcal{F}$. Prove that the topology of X defined by $\mathcal{U}$ is the product topology. For each $\mu \in M$, let $\mathcal{B}_\mu$ denote a basis of $\mathcal{U}_\mu$. Prove that the family $\{W(U_\mu) \mid \mu \in M, U_\mu \in \mathcal{B}_\mu\}$ is a sub-basis of $\mathcal{U}$. Hence deduce that the uniform product of a countable family of pseudo-metrizable uniform spaces is pseudo-metrizable.

3. UNIFORMIZATION OF TOPOLOGICAL SPACES

A topological space X is said to be *uniformizable* iff there exists a uniformity $\mathcal{U}$ in X which defines the topology τ to X .

THEOREM 3.1. *A topological space X is uniformizable iff it is completely regular.*

Proof. Necessity. Assume that X is uniformizable and let $\mathcal{U}$ denote a uniformity in X which defines the topology τ of the space X . To prove the complete regularity of X , let us consider an arbitrary point $p \in X$ and any neighborhood W of p in X . It suffices to establish the existence of a map $\chi: X \rightarrow I$ from the space X into the closed unit interval I of real numbers satisfying $\chi(p) = 0$ and $\chi(q) = 1$ for every point $q \in X \setminus W$.

Since $\mathcal{U}$ defines the topology τ of X and W is a neighborhood of p in

X , there exists a member U of $\mathcal{U}$ satisfying $U(p) \subset W$. By a simple inductive construction based on (1.2), we can construct for every non-negative integer n a symmetric member $U_n \subset U$ of $\mathcal{U}$ satisfying

$$U_n \circ U_n \subset U_{n-1} \subset U$$

whenever n is positive.

Next, we shall construct a member $V_r \in \mathcal{U}$ for every dyadic rational number r satisfying $0 < r \leq 1$. Such a dyadic rational number r can be uniquely expressed as a finite sum of the form

$$r = \sum_{i=1}^m 2^{-n_i}$$

with $0 \leq n_1 < n_2 < \cdots < n_m$. Define

$$V_r = U_{n_1} \circ U_{n_2} \circ \cdots \circ U_{n_m}.$$

For convenience, we also define $V_0 = \Delta$ although it is not necessarily a member of $\mathcal{U}$. Besides, we should observe that $V_1 = U_0$ holds.

Now let us establish by induction that the inclusions

$$V_{k2^{-n}} \subset V_{k2^{-n}} \circ U_n \subset V_{(k+1)2^{-n}}$$

hold for every nonnegative integer n and all integers $k = 0, 1, \dots, 2^n - 1$. For $n = 0$, the inclusions are true because of the following relations

$$V_0 = \Delta \subset V_0 \circ U_0 = \Delta \circ U_0 = U_0 = V_1.$$

For the general inductive step, let $n > 0$ and assume that the inclusions hold for $n - 1$. We will establish the inclusions for n . Since

$$V_{k2^{-n}} = V_{k2^{-n}} \circ \Delta \subset V_{k2^{-n}} \circ U_n$$

is obviously true, it remains to prove the inclusion

$$V_{k2^{-n}} \circ U_n \subset V_{(k+1)2^{-n}}$$

for all integers $k = 0, 1, \dots, 2^n - 1$.

If k is an even integer, say $k = 2j$, we have

$$k2^{-n} = (2j)2^{-n} = j2^{-(n-1)},$$

$$(k+1)2^{-n} = (2j+1)2^{-n} = j2^{-(n-1)} + 2^{-n}.$$

According to our definition of V_r , this implies

$$V_{(k+1)2^{-n}} = V_{j2^{-(n-1)}} \circ U_n = V_{k2^{-n}} \circ U_n.$$

Hence the inclusion is proved for this case.

If k is an odd integer, say $k = 2j + 1$, we have

$$\begin{aligned} k2^{-n} &= (2j + 1)2^{-n} = j2^{-(n-1)} + 2^{-n}, \\ (k + 1)2^{-n} &= (2j + 2)2^{-n} = (j + 1)2^{-(n-1)}. \end{aligned}$$

According to our inductive hypothesis, we have

$$V_{j2^{-(n-1)}} \circ U_{n-1} \subset V_{(j+1)2^{-(n-1)}}.$$

By the aid of the inclusion $U_n \circ U_n \subset U_{n-1}$ and our definition of V_r , we obtain

$$\begin{aligned} V_{k2^{-n}} \circ U_n &= V_{j2^{-(n-1)}+2^{-n}} \circ U_n = V_{j2^{-(n-1)}} \circ U_n \circ U_n \\ &\subset V_{j2^{-(n-1)}} \circ U_{n-1} \subset V_{(j+1)2^{-(n-1)}} = V_{(k+1)2^{-n}}. \end{aligned}$$

This completes the inductive proof of the inclusions.

As a consequence of these inclusions, we have the inclusion

$$V_r \subset V_s$$

for any two dyadic rational numbers r and s satisfying $0 \leq r \leq s \leq 1$. In fact, there exists a positive integer n such that $r = i2^{-n}$ and $s = j2^{-n}$ hold with integers i and j satisfying

$$0 \leq i \leq j \leq 2^n.$$

Hence we have

$$V_r = V_{i2^{-n}} \subset V_{(i+1)2^{-n}} \subset \cdots \subset V_{j2^{-n}} = V_s.$$

Now we define a function $\chi: X \rightarrow I$ by taking

$$\chi(x) = \begin{cases} \sup \{r \mid x \notin V_r(p)\}, & (\text{if } x \neq p), \\ 0, & (\text{if } x = p). \end{cases}$$

where the supremum is taken over all dyadic rational numbers in the closed unit interval I . By definition, we have $\chi(p) = 0$. On the other hand, let q denote any point in the set $X \setminus W$. Because of the inclusions $U(p) \subset W$ and $V_1 = U_0 \subset U$, we have $q \notin V_1(p)$. By definition, this implies $\chi(q) = 1$.

It remains to establish the continuity of this function χ . For this purpose, let r denote any real number in I . According to the implication (iv) $\Rightarrow$ (i) in (II, 3.1), it suffices to prove that the following two subsets

$$\begin{aligned} A &= \{x \in X \mid \chi(x) < r\} \\ B &= \{x \in X \mid \chi(x) > r\} \end{aligned}$$

of the space X are open.

To prove that A is open, let x denote an arbitrary point in A . Then we have

$$\chi(x) = s < r \leq 1.$$

Let $t = r - s > 0$. Choose a positive integer $n > 2/t$. Since $\mathcal{U}$ defines the topology τ of X , it suffices to prove the inclusion $U_n(x) \subset A$. For this purpose, let k denote the uniquely determined positive integer satisfying

$$(k-1)2^{-n} \leq s < k2^{-n}.$$

Since $\chi(x) < k2^{-n}$ holds, it follows from the definition of χ that $x \in V_{k2^{-n}}(p)$. Let y denote any point of $U_n(x)$. Hence we obtain $(p, x) \in V_{k2^{-n}}$ and $(x, y) \in U_n$. It follows that

$$(p, y) \in V_{k2^{-n}} \circ U_n \subset V_{(k+1)2^{-n}}.$$

Therefore, we have $y \in V_{(k+1)2^{-n}}(p)$. According to the definition of the function χ , this implies $\chi(y) \leq (k+1)2^{-n}$. Hence we obtain

$$\chi(y) - s \leq (k+1)2^{-n} - (k-1)2^{-n} = 2(2^{-n}) < t.$$

This implies $\chi(y) < s + t = r$. Thus we have proved $y \in A$. This implies $U_n(x) \subset A$ and hence A is an open subset of X .

To prove that B is open, let x denote an arbitrary point in B . Then we have

$$\chi(x) = s > r \geq 0.$$

Let $t = s - r > 0$. Choose a positive integer $n > 2/t$. It remains to prove $U_n(x) \subset B$. For this purpose, let k denote the uniquely determined positive integer satisfying

$$(k-1)2^{-n} \leq r < k2^{-n}.$$

Let y denote any point of $U_n(x)$. To prove $y \in B$ by contradiction, assume $y \notin B$. Then we have $\chi(y) \leq r < k2^{-n}$. By the definition of χ , this implies $y \in V_{k2^{-n}}(p)$. Thus we have $(p, y) \in V_{k2^{-n}}$. Since U_n is symmetric and y is in $U_n(x)$, we have $(y, x) \in U_n$. It follows that

$$(p, x) \in V_{k2^{-n}} \circ U_n \subset V_{(k+1)2^{-n}}.$$

Hence we have $x \in V_{(k+1)2^{-n}}$. According to the definition of χ , this implies $\chi(x) \leq (k+1)2^{-n}$. Hence we obtain

$$s - r = \chi(x) - r \leq (k+1)2^{-n} - (k-1)2^{-n} = 2(2^{-n}) < t.$$

This contradicts the equality $s - r = t$ and proves $y \in B$. Hence $U_n(x) \subset B$ and B is an open subset of X . Now the necessity proof is complete.

Sufficiency. Let X denote any completely regular space. We shall construct a uniformity $\mathcal{U}$ in X which defines the topology τ of X . For this purpose, let us consider the family $\mathcal{F}$ of all maps from X into the closed unit interval I of real numbers.

For every member $f: X \rightarrow I$ of $\mathcal{F}$ define a function $d_f: X^2 \rightarrow R$ by taking

$$d_f(a, b) = |f(a) - f(b)|$$

for every point (a, b) of X^2 . Since the conditions (M1) and (M2) in (IV, § 1) are obviously satisfied, d_f is a pseudo-metric in the set X and hence defines a uniformity $\mathcal{U}_f$ in X according to the example (1) of § 1.

Consider the family $\mathcal{S} = \bigcup \{\mathcal{U}_f \mid f \in \mathcal{F}\}$ of subsets of X^2 . Since each $\mathcal{U}_f$ is a uniformity in X , $\mathcal{S}$ obviously satisfies the conditions (a), (b), and (c) in Exercise 1F. Hence $\mathcal{S}$ is a sub-basis of a uniquely determined uniformity $\mathcal{U}$ in the set X .

It remains to prove that $\mathcal{U}$ defines the topology τ of X . For this purpose, let σ denote the topology in X defined by $\mathcal{U}$. We shall prove $\sigma = \tau$.

To prove $\sigma \subset \tau$, let G denote any open set in σ . To prove that G is also in τ , let x denote an arbitrary point in G . Then there exists a member U of $\mathcal{U}$ satisfying $U(x) \subset G$. Since $\mathcal{S}$ is a sub-basis of $\mathcal{U}$, there exists a finite subfamily $\{U_1, \dots, U_n\}$ of $\mathcal{S}$ satisfying

$$U_1 \cap \dots \cap U_n \subset U.$$

By the definition of $\mathcal{S}$, there exist for each integer $i = 1, \dots, n$ a map $f_i: X \rightarrow I$ and a positive real number r_i satisfying

$$U_i = \{(a, b) \in X^2 \mid |f_i(a) - f_i(b)| < r_i\}.$$

Since $f_1, \dots, f_n$ are continuous, there exists a neighborhood W of x in the topology τ such that

$$|f_i(x) - f_i(y)| < r_i$$

holds for every point $y \in W$ and all integers $i = 1, \dots, n$. This implies

$$W \subset U_1(x) \cap \dots \cap U_n(x) \subset U(x) \subset G$$

Hence G is an open set in τ . This proves $\sigma \subset \tau$.

To prove $\tau \subset \sigma$, let V denote any open set in τ . To prove that V is also in σ , let p denote an arbitrary point in V . Since X is completely regular, there exists a member $f: X \rightarrow I$ of $\mathcal{F}$ satisfying $f(p) = 0$ and $f(q) = 1$ for every $q \in X \setminus V$. Consider the member U of $\mathcal{S}$ defined by

$$U = \{(a, b) \in X^2 \mid |f(a) - f(b)| < 1\}.$$

Then we have

$$U(p) = \{x \in X \mid f(x) < 1\} \subset V.$$

This proves that V is an open set in σ . Hence $\tau \subset \sigma$ holds.

Consequently, we have proved $\sigma = \tau$ and completed the proof. $\parallel$

COROLLARY 3.2. *Every topological group is completely regular.*

COROLLARY 3.3. *A uniform space X is a Tychonoff space iff its uniformity $\mathcal{U}$ satisfies $\bigcap \{U \mid U \in \mathcal{U}\} = \Delta$.*

EXERCISES

- 3A. Prove that the uniformity $\mathcal{U}$ of a compact uniform space $(X, \mathcal{U})$ consists of all neighborhoods of the diagonal Δ in X^2 and hence is completely determined by the topology τ of X .
- 3B. Let F denote a family of covers of a given set X satisfying the conditions:
- (a) Any two members of F have a common refinement in F .
 - (b) Every member of F has a star refinement in F .
 - (c) Every cover of X which has a refinement in F belongs to F .
- For each member $\gamma \in F$, let

$$U_\gamma = \bigcup \{C \times C \mid C \in \gamma\} \subset X^2$$

and let $\mathcal{B}_F = \{U_\gamma \mid \gamma \in F\}$. Prove that $\mathcal{B}_F$ is a basis of a uniquely determined uniformity $\mathcal{U}_F$ in X called the *uniformity defined by the family F* .

- 3C. A cover γ of a uniform space $(X, \mathcal{U})$ is said to be *uniform* iff there exists a member U of the uniformity $\mathcal{U}$ such that, for every point $x \in X$, $U(x)$ is contained in a member of the cover γ . Prove that the family F in exercise 3B is precisely the family of all uniform covers of the uniform space $(X, \mathcal{U}_F)$.
- 3D. Prove that every open cover of a compact uniform space is a uniform cover.

4. UNIFORM PROPERTIES

Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ denote arbitrarily given uniform spaces. A function

$$f: X \rightarrow Y$$

is said to be *uniformly continuous* (with respect to the uniformities $\mathcal{U}$ and $\mathcal{V}$) iff, for every member $V \in \mathcal{V}$, there exists a member $U \in \mathcal{U}$ satisfying

$$(f(a), f(b)) \in V$$

for every point $(a, b) \in U$.

For example, consider pseudo-metric spaces (X, d) and (Y, e) . Let $\mathcal{U}$ and $\mathcal{V}$ denote the uniformities defined by d and e respectively. Then one can easily verify that a function $f: X \rightarrow Y$ is uniformly continuous with respect to the uniformities $\mathcal{U}$ and $\mathcal{V}$ iff it is uniformly continuous with respect to the pseudo-metrics d and e .

Consider the Cartesian square

$$F = f \times f: X^2 \rightarrow Y^2$$

of a given function $f: X \rightarrow Y$ defined by

$$F(a, b) = (f(a), f(b))$$

for every point (a, b) of X^2 . The proof of the following proposition is straightforward and hence left to the student.

PROPOSITION 4.1. *For an arbitrary function $f: X \rightarrow Y$ from a uniform space $(X, \mathcal{U})$ into a uniform space $(Y, \mathcal{V})$, the following five statements are equivalent:*

- (i) *The function f is uniformly continuous.*
- (ii) *For every $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ satisfying $F(U) \subset V$.*
- (iii) *For every $V \in \mathcal{V}$, $F^{-1}(V)$ is a member of $\mathcal{U}$.*
- (iv) *There exists a basis $\mathcal{B}$ of $\mathcal{V}$ such that $F^{-1}(V) \in \mathcal{U}$ holds for every $V \in \mathcal{B}$.*
- (v) *There exists a sub-basis $\mathcal{S}$ of $\mathcal{V}$ such that $F^{-1}(V) \in \mathcal{U}$ holds for every $V \in \mathcal{S}$.*

THEOREM 4.2. *Every uniformly continuous function $f: X \rightarrow Y$ from a uniform space $(X, \mathcal{U})$ into a uniform space $(Y, \mathcal{V})$ is continuous.*

Proof. To prove the continuity of f , let H denote any open set of Y . It suffices to prove that the inverse image

$$G = f^{-1}(H)$$

is an open set of X . For this purpose, let x denote any point in G . Then the point $y = f(x)$ is in the open set H of Y . Since $\mathcal{V}$ defines the topology of Y , there exists a member $V \in \mathcal{V}$ satisfying

$$V(y) \subset H.$$

Since f is uniformly continuous, it follows that there exists a member $U \in \mathcal{U}$ such that

$$(f(a), f(b)) \in V$$

holds for every point $(a, b) \in U$. This implies

$$f[U(x)] \subset V(y) \subset H$$

and hence

$$U(x) \subset f^{-1}(H) = G.$$

This proves that G is an open set of X . $\parallel$

By a *uniform equivalence* from a uniform space $(X, \mathcal{U})$ into a uniform space $(Y, \mathcal{V})$, we mean a bijective uniformly continuous function $f: X \rightarrow Y$ such that its inverse function $g = f^{-1}: Y \rightarrow X$ is also uniformly continuous. Thus every uniform equivalence is a homeomorphism and every bijective isometric function of pseudo-metric spaces is a uniform equivalence with respect to the uniformities defined by their pseudo-metrics.

A property of a uniform space is called a *uniform property* iff it is preserved by every uniform equivalence. Hence every topological property of a uniform space is a uniform property and every uniform property of a pseudo-metric space is a metric property. As we shall see, the converses are both false.

PROPOSITION 4.3. *Boundedness of a pseudo-metric space is not a uniform property.*

Proof. Consider any pseudo-metric space (X, d) and define a function $d': X^2 \rightarrow R$ by taking

$$d'(a, b) = \min \{1, d(a, b)\}$$

for every point (a, b) of X^2 . In the proof of (IV, 1.5), it was proved that d' is a pseudo-metric in X and defines the same topology τ of X as does d .

We will prove that d and d' define the same uniformity in X . For this purpose, let $\mathcal{U}$ and $\mathcal{U}'$ denote the uniformities in X defined by d and d' respectively. We are going to prove $\mathcal{U} = \mathcal{U}'$.

To prove $\mathcal{U}' \subset \mathcal{U}$, let U' denote any member of $\mathcal{U}'$. Since d' defines $\mathcal{U}'$, there exists a positive real number r satisfying

$$d'^{-1}(r) \subset U'.$$

Since $d'(a, b) \leq d(a, b)$ holds for all points (a, b) of X^2 , we have

$$d^{-1}(r) \subset d'^{-1}(r) \subset U'.$$

Since d defines $\mathcal{U}$, this implies $U' \in \mathcal{U}$. This proves $\mathcal{U}' \subset \mathcal{U}$.

To prove $\mathcal{U} \subset \mathcal{U}'$, let U denote any member of $\mathcal{U}$. Since d defines $\mathcal{U}$, there exists a positive real number $r < 1$ such that

$$d^{-1}(r) \subset U$$

holds. Since $r < 1$, it follows from the definition of d' that

$$d'^{-1}(r) = d^{-1}(r) \subset U.$$

Since d' defines $\mathcal{U}'$, this implies $U \in \mathcal{U}'$. This proves $\mathcal{U} \subset \mathcal{U}'$.

Thus we have proved $\mathcal{U} = \mathcal{U}'$. This implies that the identity function

$$i: (X, d) \rightarrow (X, d')$$

is a uniform equivalence. Assume that the given pseudo-metric space (X, d) is not bounded, say, the Euclidean n -space R^n with its Euclidean metric. Now, since the pseudo-metric space (X, d') is bounded, this proves that boundedness is not a uniform property. ||

Since boundedness is obviously a metric property, (4.3) shows that not every metric property is a uniform property.

A uniform space $(X, \mathcal{U})$ is said to be *totally bounded* (or *precompact*) iff, for an arbitrary member U of $\mathcal{U}$, there exists a finite number of points of X , say $x_1, \dots, x_n$, satisfying

$$X = U(x_1) \cup \dots \cup U(x_n).$$

To see the relation with the concept of total boundedness for pseudo-metric spaces defined in (IV, § 5) one can easily verify that a pseudo-metric space X is totally bounded with respect to its pseudo-metric d iff it is totally bounded with respect to the uniformity $\mathcal{U}$ defined by d . Hence our present definition of total boundedness is a generalization of our previous definition of the same term for pseudo-metric spaces.

PROPOSITION 4.4. *Total boundedness of a uniform space is a uniform property.*

Proof. Consider any uniform equivalence

$$f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$$

and assume that $(X, \mathcal{U})$ is totally bounded. It suffices to prove that $(Y, \mathcal{V})$ is totally bounded. For this purpose, let V denote an arbitrary member of $\mathcal{V}$. Since f is uniformly continuous, there exists a member U of $\mathcal{U}$ such that

$$(f(a), f(b)) \in V$$

holds for all points (a, b) in U . Since $(X, \mathcal{U})$ is totally bounded, there exists a finite number of points of X , say $x_1, \dots, x_n$, satisfying

$$X = U(x_1) \cup \dots \cup U(x_n).$$

For each $i = 1, \dots, n$, let $y_i = f(x_i) \in Y$. It remains to prove

$$Y = V(y_1) \cup \dots \cup V(y_n).$$

For this purpose, let y denote any point in Y . Since f is surjective, there is a point x in X with $y = f(x)$. Choose an integer i , $1 \leq i \leq n$, satisfying $x \in U(x_i)$. Then it follows that $y \in V(y_i)$. This completes the proof. ||

At first one might be surprised that total boundedness is a uniform property while boundedness is not. However, a second look at their definitions for pseudo-metric spaces will show that there is nothing strange in this. For boundedness is an "upper-end property" of the pseudo-metric while total boundedness is a "lower-end property," as described at the beginning of the present chapter.

By a *Cauchy net* (or *fundamental net*) in a uniform space $(X, \mathcal{U})$, we mean a net

$$\phi: D \rightarrow X$$

in the space X such that, for an arbitrary member U of $\mathcal{U}$, there exists a residual subset E of D satisfying

$$(\phi(a), \phi(b)) \in U$$

for any two members a and b of E . As in (IV, 7.1), one can prove that every convergent net in a uniform space is a Cauchy net.

A uniform space $(X, \mathcal{U})$ is said to be *complete* iff every Cauchy net in X converges to a point in X . It is well-known that a pseudo-metric space X is complete with respect to its pseudo-metric d and its Cauchy sequences iff X is complete with respect to the uniformity $\mathcal{U}$ defined by d and the Cauchy nets, [K, p. 193]. Hence our present definition of completeness is a generalization of our previous definition of the same term for pseudo-metric spaces.

PROPOSITION 4.5. *Completeness of a uniform space is a uniform property.*

Proof. Consider any uniform equivalence

$$f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$$

and assume that $(X, \mathcal{U})$ is complete. It suffices to prove that $(Y, \mathcal{V})$ is complete. For this purpose, let $\phi: D \rightarrow Y$ denote any Cauchy net in Y .

Since f is a uniform equivalence, we have a uniformly continuous inverse function

$$g = f^{-1}: (Y, \mathcal{V}) \rightarrow (X, \mathcal{U}).$$

Then the composed function

$$\psi = g \circ \phi: D \rightarrow X$$

is a net in X satisfying $\phi = f \circ \psi$.

To prove that ψ is a Cauchy net in $(X, \mathcal{U})$, let U denote any member of $\mathcal{U}$. Since g is uniformly continuous, there is a member V of $\mathcal{V}$ such that

$$(g(p), g(q)) \in U$$

holds for every point (p, q) of V . Since ϕ is a Cauchy net in $(Y, \mathcal{V})$, there exists a residual subset E of D such that

$$(\phi(a), \phi(b)) \in V$$

holds for all members a and b of E . These facts imply that

$$(\psi(a), \psi(b)) = (g[\phi(a)], g[\phi(b)]) \in U$$

holds for all members a and b of E . Hence ψ is a Cauchy net in X .

Since $(X, \mathcal{U})$ is complete, ψ converges to a point x in X . Denote $y = f(x)$. It remains to prove that ϕ converges to y . For this purpose, let M denote an arbitrary neighborhood of y in Y . Since $\mathcal{V}$ defines the topology of Y , there exists a member V of $\mathcal{V}$ satisfying $V(y) \subset M$. Since f is uniformly continuous, there exists a member U of $\mathcal{U}$ such that

$$(f(p), f(q)) \in V$$

holds for every point (p, q) of U . Since ψ converges to x , there exists a residual subset E of D such that $\psi(e) \in U(x)$ holds for all members e of E . Hence we obtain

$$\phi(e) = f[\psi(e)] \subset f[U(x)] \subset V(y) \subset M$$

for all members e of E . This proves that ϕ converges to the point y . ||

Consider the Euclidean n -space R^n and its open unit ball B^n . Since B^n is totally bounded while R^n is not, total boundedness is not a topological property. On the other hand, since R^n is complete while B^n is not, completeness is not a topological property. These examples show that not every uniform property of a uniform space is a topological property.

Generalizing the proof of (IV, 7.10) by using nets instead of sequences one can establish the following theorem.

THEOREM 4.6. *A uniform space $(X, \mathcal{U})$ is compact iff it is complete and totally bounded.*

Here, the conjunction of two non-topological uniform properties is equivalent to a topological property.

EXERCISES

- 4A. Let $f: X \rightarrow Y$ denote an arbitrary function from a set X into a uniform space $(Y, \mathcal{V})$. Prove that there exists a smallest uniformity $\mathcal{U}$ in X which makes f uniformly continuous. This uniformity $\mathcal{U}$ in X is called the *uniformity induced* by f and $\mathcal{V}$ and will be denoted by $f^*(\mathcal{V})$. In particular, if f is the inclusion function of a subset X of Y into Y , then the uniformity $\mathcal{U} = f^*(\mathcal{V})$ is called the *relative uniformity* in X or the *restriction* of $\mathcal{V}$ on X .
- 4B. Let $\mathcal{F}$ denote any family of functions $f: X \rightarrow (Y_f, \mathcal{V}_f)$ from a given set X into uniform spaces $(Y_f, \mathcal{V}_f)$. Prove that there exists a smallest uniformity $\mathcal{U}$ in X which makes all functions $f \in \mathcal{F}$ uniformly continuous. This uniformity $\mathcal{U}$ is called the *uniformity induced by the family* $\mathcal{F}$. In particular, let X denote the Cartesian product of the family $\{Y_f \mid f \in \mathcal{F}\}$ and f denote the natural projection. In this case, the uniformity $\mathcal{U}$ in X induced by the family $\mathcal{F}$ is precisely the product uniformity in X . Hence the natural projections of a product uniform space are uniformly continuous.
- 4C. Prove that a function $f: X \rightarrow Y = \prod_{\mu} Y_{\mu}$ from a uniform space X into the product uniform space Y of any family $\mathcal{F} = \{Y_{\mu} \mid \mu \in M\}$ of uniform spaces Y_{μ} is uniformly continuous iff, for each $\mu \in M$, the composition

$$p_{\mu} \circ f: X \rightarrow Y_{\mu}$$

of f and the natural projection p_{μ} is uniformly continuous.

- 4D. Let $(X, \mathcal{U})$ denote any uniform space. A pseudo-metric $d: X^2 \rightarrow \mathbb{R}$ in X is uniformly continuous with respect to the product uniformity in X^2 iff $d^{-1}(0, r) \in \mathcal{U}$ holds for every positive real number r . In particular, d is uniformly continuous if d defines the uniformity $\mathcal{U}$.
- 4E. Prove that the uniformity $\mathcal{U}$ of a uniform space X is defined by the family $\mathcal{F}$ of all pseudo-metrics in X which are uniformly continuous on the product uniform space X .

- 4F. Prove that every uniform space is uniformly equivalent to a subspace of a product of pseudo-metrizable uniform spaces and that every uniform Fréchet space is uniformly equivalent to a subspace of a product of metrizable uniform spaces.
- 4G. Prove that a topological space X is completely regular iff it is imbeddable into a topological product of pseudo-metrizable spaces and that X is a Tychonoff space iff it is imbeddable into a topological product of metrizable spaces.
- 4H. Prove that every continuous function $f: X \rightarrow Y$ from a compact uniform space X into any uniform space Y is uniformly continuous.
- 4I. Prove that every closed subspace (with relative uniformity) of any complete uniform space is complete and every complete subspace of a uniform Fréchet space is closed.
- 4J. Prove that the product uniform space X of any family $\mathcal{F} = \{X_\mu \mid \mu \in M\}$ of uniform spaces is complete iff every X_μ is complete.
- 4K. Prove that every uniformly continuous function $f: E \rightarrow Y$ from a dense subspace E of a uniform space X into a complete uniform Fréchet space Y has a unique continuous extension $f^*: X \rightarrow Y$. Also prove that f^* is uniformly continuous.
- 4L. *Completions of uniform spaces.* Prove that every uniform space X is uniformly equivalent to a dense subspace of a complete uniform space X^* and that every uniform Fréchet space X is uniformly equivalent to a dense subspace of a complete uniform Fréchet space X^* . For the latter, prove that X^* is uniquely determined up to a uniform equivalence.

5. AXIOMATIC APPROACH TO BOUNDEDNESS

By a *boundedness* in a topological space X , we mean a nonempty family $\mathcal{B}$ of subsets of X , called the *bounded sets* of X , satisfying the following two conditions:

- (B1) Every subset of a bounded set is bounded.
- (B2) The union of a finite number of bounded sets is bounded.

The following two statements (B3) and (B4) are immediate consequences of (B1).

- (B3) The empty subset $\square$ of X is bounded.
- (B4) The intersection of a nonempty family of bounded sets is bounded.

By a *universe*, we mean a topological space X together with a given boundedness $\mathcal{B}$ in X . A universe X is said to be *bounded*, iff the whole space X is bounded and hence every subset of X is bounded.

The following proposition is obvious.

PROPOSITION 5.1. *If a universe X with boundedness $\mathcal{B}$ is not bounded, then the family*

$$\mathcal{C} = \{X \setminus B \mid B \in \mathcal{B}\}$$

of the complements $X \setminus B$ has the following properties:

- (C1) *None of the sets $X \setminus B$ is bounded.*
- (C2) *Every subset of X which contains a member of $\mathcal{C}$ is itself a member of $\mathcal{C}$.*
- (C3) *The intersection of a finite number of members of $\mathcal{C}$ is a member of $\mathcal{C}$.*

Any family $\mathcal{C}$ of subsets of X satisfying (C2) and (C3) is called a *filter* in X . Hence the family $\mathcal{C}$ in (5.1) will be referred to as the *filter at infinity* of the universe X .

EXAMPLES OF BOUNDEDNESSES:

(1) *The extreme boundednesses.* Let X denote an arbitrary topological space. The family $\mathcal{S} = \{\square\}$ which consists of the empty subset $\square$ of X only is a boundedness in X ; in fact, it is the *smallest boundedness* in X . On the other hand, the family $\mathcal{L} = 2^X$ of all subsets of X is a boundedness in X , in fact, it is the *largest boundedness* in X . The universe $(X, \mathcal{L})$ is bounded.

(2) *The compact boundedness.* Let X denote an arbitrary topological space. Consider the family $\mathcal{K}$ of subsets of X defined as follows: a subset B of X is in $\mathcal{K}$ iff its closure $\text{Cl}(B)$ in X is compact. By the properties of closures and compact subsets, one can easily verify that $\mathcal{K}$ is a boundedness in X , called the *compact boundedness* in X . Obviously, the universe $(X, \mathcal{K})$ is bounded iff X is compact.

(3) *Boundedness defined by a pseudo-metric.* Let $d: X^2 \rightarrow R$ denote any pseudo-metric defined in an arbitrarily given topological space X . Define a family $\mathcal{B}_d$ of subsets of X as follows: a subset B of X is in $\mathcal{B}_d$ iff there exists a positive real number r such that $d(a, b) < r$ holds for all points a and b in B . By means of the property (M1) of a pseudo-metric d , one can easily verify that $\mathcal{B}_d$ is a boundedness in X called the *boundedness defined by the pseudo-metric d* . In particular, if X is a pseudo-metric space with d as its pseudo-metric, then $(X, \mathcal{B}_d)$ will be called a *pseudo-metric universe*. Obviously, the universe $(X, \mathcal{B}_d)$ is bounded iff d is bounded.

(4) *Boundedness defined by a uniformity.* Let $\mathcal{U}$ denote an arbitrary uniformity in any given topological space X . Define a family $\mathcal{B}_{\mathcal{U}}$ of subsets of X as follows: a subset B of X is in $\mathcal{B}_{\mathcal{U}}$ iff, for every member U of $\mathcal{U}$, there exist a positive integer n and a finite subset F of X satisfying $B \subset U^n(F)$, where U^n is defined inductively by $U^1 = U$ and

$$U^n = U^{n-1} \circ U$$

for every integer $n > 1$. One can easily verify that $\mathcal{B}_{\mathcal{U}}$ is a boundedness in X , called the *boundedness defined by the uniformity $\mathcal{U}$* . In particular, if X is a uniform space with $\mathcal{U}$ as its uniformity, then $(X, \mathcal{B}_{\mathcal{U}})$ will be called a *uniform universe*.

To illustrate possible relations between these examples, let us consider the Euclidean n -space

$$X = R^n$$

with $d: X^2 \rightarrow R$ denoting the Euclidean metric. Let $d': X^2 \rightarrow R$ denote the bounded metric defined by

$$d'(a, b) = \min \{1, d(a, b)\}$$

for every point (a, b) of X^2 . In the proof of (4.3), we proved that d and d' define the same uniformity $\mathcal{U}$ in X . One can easily verify the following relations:

$$\mathcal{K} = \mathcal{B}_{\mathcal{U}} = \mathcal{B}_d \neq \mathcal{B}_{d'} = \mathcal{L}.$$

The Euclidean n -space R^n together with the boundedness $\mathcal{K} = \mathcal{B}_{\mathcal{U}} = \mathcal{B}_d$ is called the *Euclidean n -universe*.

A point x of a universe X is said to be a *finite point* iff it has a bounded neighborhood in X ; otherwise, x is called a *point at infinity*. A universe X is said to be *locally bounded* iff every point of X is finite. For example, every pseudo-metric universe is locally bounded.

The following two propositions are obvious.

PROPOSITION 5.2. *Every compact subset of a locally bounded universe is bounded.*

PROPOSITION 5.3. *The compact boundedness in a Hausdorff space X makes X a locally bounded universe iff X is locally compact.*

A universe X is said to be *boundedly compact* iff every bounded closed set of X is compact.

The following proposition is obvious.

PROPOSITION 5.4. *A pseudo-metric universe is boundedly compact iff every closed ball is compact.*

COROLLARY 5.5. *Every boundedly compact pseudo-metric universe is locally compact.*

For example, the Euclidean n -universe R^n is boundedly compact. On the other hand, the *Hilbert universe* H , which is the Hilbert space together with the boundedness defined by its metric, fails to be boundedly compact.

A boundedness $\mathcal{B}$ in a topological space X is said to be *closed* iff the closure of every bounded set is bounded; $\mathcal{B}$ is said to be *open* iff every bounded set is contained in some open bounded set. By a *proper boundedness* in X , we mean a boundedness $\mathcal{B}$ which is both closed and open.

By a *basis* of a boundedness $\mathcal{B}$ in a topological space X , we mean a subfamily $\mathcal{C}$ of $\mathcal{B}$ such that every member of $\mathcal{B}$ is contained in some member of $\mathcal{C}$. Obviously, the boundedness $\mathcal{B}$ is completely determined by any of its bases.

By a *closed basis* of a boundedness $\mathcal{B}$ in X , we mean a basis $\mathcal{C}$ of $\mathcal{B}$ such that every member of $\mathcal{C}$ is a closed subset of X . Similarly, an *open basis* of a boundedness $\mathcal{B}$ in X is a basis $\mathcal{C}$ of $\mathcal{B}$ such that every member of $\mathcal{C}$ is an open subset of X . Clearly, a boundedness $\mathcal{B}$ in X is closed iff it has a closed basis and is open iff it has an open basis.

By a *sub-basis* of a boundedness $\mathcal{B}$ in X , we mean a subfamily $\mathcal{D}$ of $\mathcal{B}$ such that the family $\mathcal{C}$ which consists of the unions of the members of the finite subfamilies of $\mathcal{D}$ is a basis of $\mathcal{B}$. Clearly every family $\mathcal{F}$ of subsets of X is a sub-basis of some boundedness $\mathcal{B}$ in X , called the *boundedness generated by $\mathcal{F}$* .

PROPOSITION 5.6. *The boundedness of any pseudo-metric universe is proper and has a countable basis.*

Proof. Consider an arbitrary pseudo-metric universe X with its topology τ and its boundedness $\mathcal{B}$ defined by a pseudo-metric $d: X^2 \rightarrow R$. Since the proposition is trivial in case X is empty, we assume that X is nonempty.

Select a point $z \in X$ as our base point. For each positive integer n , consider the open ball B_n and the closed ball C_n defined by

$$B_n = \{x \in X \mid d(z, x) < n\}, \quad C_n = \{x \in X \mid d(z, x) \leq n\}.$$

Since d defines both the topology τ and the boundedness $\mathcal{B}$ of X , it is easy to verify that the family $\{B_n \mid n \in N\}$ is an open basis of $\mathcal{B}$ and the family

$\{C_n \mid n \in N\}$ is a closed basis of $\mathcal{B}$. This proves that $\mathcal{B}$ is proper and has a countable basis. $\parallel$

A bounded set B_* of a boundedness $\mathcal{B}$ in a topological space X is said to be *maximal* iff every $B \in \mathcal{B}$ is a subset of B_* .

PROPOSITION 5.7. *For an arbitrary boundedness $\mathcal{B}$ in a topological space X , the following three statements are equivalent:*

- (a) $\mathcal{B}$ has a maximal bounded set B_* .
- (b) $\mathcal{B}$ has a finite basis.
- (c) The union of any collection of bounded sets is bounded.

Proof. (a) $\Rightarrow$ (b). The maximal bounded set B_* itself forms a basis of $\mathcal{B}$.

(b) $\Rightarrow$ (c). Assume that $\mathcal{B}$ admits a finite basis $\mathcal{A} = \{A_1, \dots, A_n\}$. Then the set

$$A = A_1 \cup \dots \cup A_n$$

is bounded and forms a basis of $\mathcal{B}$. Since every bounded set is a subset of A , the union of any collection of bounded sets is still a subset of A and hence is bounded.

(c) $\Rightarrow$ (a). Let B_* denote the union of all members of $\mathcal{B}$. Then B_* is bounded according to (c). Hence B_* is a maximal bounded set. $\parallel$

LEMMA 5.8. *If $\mathcal{A}$ is an arbitrary basis of a boundedness $\mathcal{B}$ with a countable basis but without maximal bounded set, then there exists a countable basis $\mathcal{C}$ of $\mathcal{B}$ which consists of a strictly increasing sequence of members of $\mathcal{A}$.*

Proof. Let $\mathcal{D} = \{D_1, D_2, \dots\}$ denote any countable basis of $\mathcal{B}$. Define a subfamily $\mathcal{C} = \{C_1, C_2, \dots\}$ of $\mathcal{A}$ as follows. Choose a $C_1 \in \mathcal{A}$ which contains D_1 . Assume that $C_1, C_2, \dots, C_n$ have been so chosen from $\mathcal{A}$ that C_i contains $C_{i-1} \cup D_i$ and is different from C_{i-1} for $i = 2, \dots, n$. Since $\mathcal{B}$ has no maximal bounded set, there exists at least one member $B_0 \in \mathcal{B}$ which is not a subset of C_n . Since $\mathcal{A}$ is a basis of $\mathcal{B}$, there exists a member C_{n+1} of $\mathcal{A}$ which contains the set $C_n \cup D_{n+1} \cup B_0$. This completes the inductive construction of the strictly increasing sequence $\mathcal{C}$.

To prove that $\mathcal{C}$ is a basis of $\mathcal{B}$, let B denote any member of $\mathcal{B}$. Since $\mathcal{D}$ is a basis of $\mathcal{B}$, there exists a positive integer n with $B \subset D_n \subset C_n$. Hence $\mathcal{C}$ is a basis of $\mathcal{B}$. $\parallel$

LEMMA 5.9. *Every proper boundedness $\mathcal{B}$ with a countable basis but without maximal bounded set admits an open basis $\mathcal{G}$ which consists of a strictly*

increasing sequence of bounded open sets $G_1, G_2, \dots, G_n, \dots$ satisfying $\text{Cl}(G_n) \subset G_{n+1}$ for every $n = 1, 2, \dots$.

Proof. By (5.8), $\mathcal{B}$ admits a basis $\mathcal{C} = \{C_1, C_2, \dots\}$ which consists of a strictly increasing sequence of bounded sets. Let G_1 be an arbitrary bounded open set. Assume that bounded open sets $G_1, G_2, \dots, G_n$ have been so chosen that G_i contains $\text{Cl}(G_{i-1}) \cup C_{i-1}$ and is different from G_{i-1} for each $i = 2, \dots, n$. Since $\mathcal{B}$ has no maximal bounded set, there exists a member $B_0 \in \mathcal{B}$ which is not a subset of G_n . Since $\mathcal{B}$ is closed, $\text{Cl}(G_n)$ is a bounded set. Since $\mathcal{B}$ is open, there exists an open bounded set G_{n+1} containing $\text{Cl}(G_n) \cup C_n \cup B_0$. Hence G_{n+1} contains $\text{Cl}(G_n) \cup C_n$ and is different from G_n . This completes the inductive construction of a strictly increasing sequence

$$\mathcal{G} = \{G_1, G_2, \dots, G_n, \dots\}$$

of bounded open sets satisfying $\text{Cl}(G_n) \subset G_{n+1}$ for every $n = 1, 2, \dots$.

It remains to prove that $\mathcal{G}$ is a basis of $\mathcal{B}$. For this purpose, let B denote any member of $\mathcal{B}$. Since $\mathcal{C}$ is a basis of $\mathcal{B}$, there exists a positive integer n satisfying $B \subset C_n \subset G_{n+1}$. Hence $\mathcal{G}$ is a basis of $\mathcal{B}$. $\parallel$

By a *characteristic function* of a universe X , we mean a real-valued nonnegative continuous function χ defined on X in such a way that a subset E of X is bounded iff the least upper bound of χ over E is finite.

A universe $(X, \mathcal{B})$ is said to be *normal* iff its topological space X is normal; $(X, \mathcal{B})$ is said to be *proper* iff its boundedness $\mathcal{B}$ is proper.

THEOREM 5.10. *A normal universe X admits a characteristic function χ iff X is proper and locally bounded, with a countable basis of its boundedness.*

Proof: Necessity. Assume that X is a universe which admits a characteristic function χ . Let $p \in X$ and denote $q = \chi(p)$. Choose a real number $b > q$. Let B_b denote the subset of X which consists of all points $x \in X$ satisfying $\chi(x) < b$. From the continuity of χ , it follows that B_b is an open set. According to the definition of a characteristic function, B_b is bounded. Since p is in B_b , X is locally bounded.

Next let B denote an arbitrary bounded set. Choose a real number b with

$$b > \sup \{\chi(x) \mid x \in B\}.$$

Then B_b is a bounded open set and $\text{Cl}(B_b)$ is a bounded closed set both containing B . Hence X is proper.

Now, for each positive integer n , let B_n denote the subset of X which consists of all points $x \in X$ satisfying $\chi(x) < n$. Then $\mathcal{C} = \{B_n \mid n \in \mathbb{N}\}$

is a sequence of bounded open sets. Let B denote an arbitrary bounded set of X and choose an integer n satisfying

$$n > \sup \{ \chi(x) \mid x \in B \}.$$

Then we have $B \subset B_n$. Hence $\mathcal{C}$ is a countable basis of the boundedness of the universe X .

Sufficiency. Assume that X is a normal universe satisfying the condition of the theorem. If X is bounded, then the constant zero function on X is a characteristic function of X . Hereafter, we assume that X is not bounded. Since X is locally bounded, this implies that X cannot have a maximal bounded set. It follows from (5.9) that the boundedness of X has a basis $\mathcal{G}$ which consists of a strictly increasing sequence of bounded open sets $G_1, G_2, \dots, G_n, \dots$, satisfying $\text{Cl}(G_n) \subset G_{n+1}$ for every $n = 1, 2, \dots$.

Since $\text{Cl}(G_n)$ and $X \setminus G_{n+1}$ are two disjoint closed sets of a normal space X , it follows from Urysohn's lemma (III, 1.6) that there exists a map $\phi_n: X \rightarrow I$ satisfying $\phi_n(x) = 0$ for every $x \in \text{Cl}(G_n)$ and $\phi_n(x) = 1$ for every $x \in X \setminus G_{n+1}$. Now define a function $\chi: X \rightarrow R$ by taking

$$\chi(x) = n - 1 + \phi_n(x)$$

for every $x \in G_{n+1} \setminus G_n$ and for every n .

The function χ is obviously real-valued and nonnegative. The continuity of χ follows from the fact that $\phi_{n-1}(x) = 0$ and $\phi_n(x) = 1$ hold for every point $x \in \text{Cl}(G_n) \setminus G_n$ and every $n > 1$. Now let E denote an arbitrary subset of X . If E is bounded, then there exists a positive integer n satisfying $E \subset G_{n+1}$. This implies $\chi(x) \leq n$ for every $x \in E$. Conversely, assume that

$$\sup \{ \chi(x) \mid x \in E \}$$

is finite. Choose a positive integer n so large that $\chi(x) < n$ holds for every $x \in E$. Then we have $E \subset G_{n+1}$ and hence E is bounded. Hence χ is a characteristic function of X . ||

As applications of this theorem (5.10), we have the following pseudometrization and metrization theorems.

A universe X is said to be *pseudo-metrizable* iff there exists a pseudo-metric

$$d: X^2 \rightarrow R$$

which defines both the topology τ and the boundedness $\mathcal{B}$ of the universe X .

THEOREM 5.11. (PSEUDO-METRIZATION THEOREM). *A universe $(X, \mathcal{B})$ is pseudo-metrizable iff the following three conditions are satisfied:*

- (a) *The topological space X is pseudo-metrizable.*
- (b) *The universe X is locally bounded.*
- (c) *The boundedness $\mathcal{B}$ is proper and has a countable basis.*

Proof: Necessity. Assume that $(X, \mathcal{B})$ is pseudo-metrizable. Then (a) and (b) obviously hold. According to (5.6), we also have (c).

Sufficiency. Assume that $(X, \mathcal{B})$ satisfies the conditions (a), (b), and (c). According to (IV, 1.5), (a) implies that there exists a bounded pseudo-metric

$$e: X^2 \rightarrow R$$

which defines the topology τ of X .

By (IV, 4.1), (a) implies that X is normal. Hence we can apply (5.10) and obtain a characteristic function

$$\chi: X \rightarrow R$$

of the universe X .

Now define a function $d: X^2 \rightarrow R$ by taking

$$d(a, b) = e(a, b) + |\chi(a) - \chi(b)|$$

for every point (a, b) of X^2 . By the properties of the pseudo-metric e and the absolute values, one can easily verify that d is a pseudo-metric in X .

To prove that d defines the topology τ of X , it suffices to show that d and e are equivalent. For this purpose, consider any point $p \in X$ and an arbitrary positive real number r . Since χ is continuous and since e defines the topology τ of X , there exists a positive real number $s < r/2$ such that

$$|\chi(p) - \chi(x)| < \frac{r}{2}$$

holds for every point $x \in X$ satisfying $e(p, x) < s$. It follows that, for every $x \in X$, $e(p, x) < s$ implies

$$d(p, x) = e(p, x) + |\chi(p) - \chi(x)|$$

$$< s + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r.$$

On the other hand, $d(p, x) < s$ obviously implies $e(p, x) < r$. According to (IV, 1.3), this proves that the pseudo-metrics d and e are equivalent.

It remains to verify that the pseudo-metric d defines the boundedness

$\mathcal{B}$. For this purpose, choose a positive real number k such that $e(a, b) < k$ holds for every point (a, b) of X^2 . Let E denote any subset of X . If E is a member of $\mathcal{B}$, then it follows from the definition of a characteristic function that

$$l = \sup \{ \chi(x) \mid x \in E \}$$

is finite. Hence we obtain

$$d(a, b) = e(a, b) + |\chi(a) - \chi(b)| < k + l$$

for all points a and b of E . This proves that E is a member of $\mathcal{B}_d$. Conversely, assume that E is a member of $\mathcal{B}_d$. Then there exists a positive real number r such that $d(a, b) < r$ holds for all points a and b of E . Without loss of generality, we may assume $E \neq \square$. Choose a point $p \in E$. Then we have

$$\chi(x) \leq \chi(p) + |\chi(p) - \chi(x)| \leq \chi(p) + d(p, x) < \chi(p) + r$$

for all points $x \in E$. This implies $E \in \mathcal{B}$. Hence we obtain $\mathcal{B} = \mathcal{B}_d$, that is, d defines the given boundedness $\mathcal{B}$ of X . $\parallel$

COROLLARY 5.12. (METRIZATION THEOREM). *A universe $(X, \mathcal{B})$ is metrizable iff the following three conditions are satisfied:*

- (a) *The topological space X is metrizable.*
- (b) *The universe X is locally bounded.*
- (c) *The boundedness $\mathcal{B}$ is proper and has a countable basis.*

EXERCISES

- 5A. Consider a Hausdorff space X and a non-isolated point p of X . Define a family $\mathcal{B}$ of subsets of X as follows: a subset B of X is a member of $\mathcal{B}$ iff B does not contain the point p . Verify that $\mathcal{B}$ is an open boundedness in X but not a closed boundedness in X .
- 5B. In the Hilbert space H , consider for each positive integer n the closed unit interval X_n on the n -th axis defined by $0 \leq x_n \leq 1$ and $x_i = 0$ ($i \neq n$). Let

$$X = \bigcup_{n=1}^{\infty} X_n.$$

Define a family $\mathcal{B}$ of subsets of X as follows: a subset B of X is a member of $\mathcal{B}$ iff its closure $\text{Cl}(B)$ in X is compact. Verify that $\mathcal{B}$ is a closed boundedness in X but not an open boundedness in X .

5C. Let $\mathcal{B}$ denote a boundedness in a topological space X with a maximal bounded set B_* . Prove the following assertions:

- (a) $\mathcal{B}$ is closed iff B_* is closed.
- (b) $\mathcal{B}$ is open iff B_* is open.
- (c) $\mathcal{B}$ is proper iff B_* is both closed and open.

Then prove that a connected proper universe whose boundedness admits a finite basis is bounded.

5D. For an arbitrary topological space X , prove that the following conditions are equivalent:

- (a) X is locally compact.
- (b) The compact boundedness in X makes X a locally bounded universe.

If, in addition, X is a Hausdorff space, then show that each of the conditions (a) and (b) is equivalent to the following condition:

- (c) The compact boundedness in X is open.

5E. Prove that a universe X is boundedly compact iff every collection of bounded closed sets in X with the finite intersection property has at least one common point.

5F. By a *bounded cover* of a subset E of a universe X , we mean a family of bounded sets of X whose union contains E . Prove that a locally bounded universe X is boundedly compact iff every bounded open cover of a bounded closed set F of X has a finite subcover of F .

5G. A function $f: X \rightarrow Y$ of a universe X into a universe Y is said to be *bounded* iff the image of every bounded set of X is a bounded set of Y ; it is said to be *bounding* iff the inverse image of every bounded set of Y is a bounded set of X . A homeomorphism $h: X \rightarrow Y$ of a universe X onto a universe Y is said to be *complete* iff h is both bounded and bounding. X and Y are said to be *completely homeomorphic* iff there exists a complete homeomorphism h of X onto Y . Prove that a universe X is completely homeomorphic with a subset of the Hilbert universe H iff X is normal, proper, locally bounded, with a countable basis for its topology and a countable basis for its boundedness.

Chapter VI: TOPOLOGICAL LINEAR SPACES

As illustrative examples of topological spaces with both a natural uniformity and a natural boundedness, we will study in the present chapter topological linear spaces and, as the important special case, normed linear spaces. Although we will consider only linear spaces over real numbers, most of the results as well as the concepts can be readily extended to linear spaces over complex numbers.

1. LINEAR SPACES

By an *Abelian group*, we mean a nonempty set X together with a function

$$+ : X^2 \rightarrow X,$$

with $+(a, b)$ denoted by $a + b$ for every point (a, b) of X^2 , such that the following four conditions are satisfied:

(A1) *Commutativity*: For any two elements a and b of X , we have

$$a + b = b + a.$$

(A2) *Associativity*: For any three elements a , b , and c of X , we have

$$(a + b) + c = a + (b + c).$$

(A3) *Existence of neutral element*: There exists an element 0 of X such that, for every element $x \in X$, we have

$$x + 0 = x = 0 + x.$$

(A4) *Existence of inverse*: For every element x of X , there exists an element $-x$ of X satisfying

$$x + (-x) = 0 = (-x) + x.$$

The function $+$ will be referred to as the *addition* of the Abelian group X and, for any two elements a and b of X , the element $a + b$ will be called the *sum* of a and b .

LEMMA 1.1. In an arbitrary Abelian group X , the neutral element 0 is unique.

Proof. Let e denote any element of X satisfying $e + x = x$ for every element x of X . Then we have

$$e = e + 0 = 0.$$

This proves the uniqueness of the neutral element 0 . $\parallel$

LEMMA 1.2. In an arbitrary Abelian group X , the inverse $-x$ of any given element $x \in X$ is unique.

Proof. Let y denote any element of X satisfying $x + y = 0$. Then we have

$$y = 0 + y = [(-x) + x] + y = (-x) + (x + y) = (-x) + 0 = -x.$$

This proves the uniqueness of the inverse $-x$ of the given element $x \in X$. $\parallel$

By a *scalar multiplication* (with real numbers as scalars) in a given Abelian group X , we mean a function

$$\mu: R \times X \rightarrow X$$

from the Cartesian product $R \times X$ of the set R of all real numbers and the set X into X , with $\mu(\alpha, x)$ denoted by αx for every point (α, x) of $R \times X$, such that the following four conditions are satisfied:

(SM1) For every $\alpha \in R$ and any two elements u and v of X , we have

$$\alpha(u + v) = \alpha u + \alpha v.$$

(SM2) For arbitrary $\alpha, \beta \in R$ and every element x of X , we have

$$(\alpha + \beta)x = \alpha x + \beta x.$$

(SM3) For arbitrary $\alpha, \beta \in R$ and every element x of X , we have

$$(\alpha\beta)x = \alpha(\beta x).$$

(SM4) For the real number 1 and every element x of X , we have

$$1x = x.$$

By a *linear space* (over real numbers), we mean an Abelian group X together with a given scalar multiplication in X . Linear spaces are also called *vector spaces*. The elements of a linear space X are usually called *vectors*.

For an arbitrary linear space X , the neutral element 0 of the Abelian group X is called the *origin* (or the *zero vector*) of the linear space X . Although the real number zero and the origin are denoted by the same symbol 0 , their meaning will be clear from the context.

LEMMA 1.3. *In an arbitrary linear space X , we have*

$$0x = 0$$

for every vector $x \in X$.

Proof. For every vector $x \in X$, we have

$$0x + x = 0x + 1x = (0 + 1)x = 1x = x.$$

Then we obtain

$$0x = 0x + 0 = 0x + [x + (-x)] = (0x + x) + (-x) = x + (-x) = 0.$$

This proves (1.3). $\parallel$

LEMMA 1.4. *In an arbitrary linear space X , we have*

$$(-1)x = -x$$

for every vector $x \in X$.

Proof. For every vector $x \in X$, we have

$$x + (-1)x = (1 - 1)x = 0x = 0.$$

According to (1.2), this implies $(-1)x = -x$. $\parallel$

LEMMA 1.5. *If α is a nonzero real number and x is an arbitrary vector of a linear space X , then $\alpha x = 0$ is equivalent to $x = 0$.*

Proof. First, assume $x = 0$. Then it follows from (1.4) that we have

$$\alpha x = \alpha 0 = \alpha[x + (-1)x] = (\alpha - \alpha)x = 0x = 0.$$

Next assume $\alpha x = 0$. Since α is nonzero, α^{-1} is a well-defined real number. Hence we have

$$x = (\alpha^{-1}\alpha)x = \alpha^{-1}(\alpha x) = \alpha^{-1}0 = 0.$$

This completes the proof of (1.5). $\parallel$

EXAMPLES OF LINEAR SPACES:

(1) *The Euclidean n -space R^n .* For any two points $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ of R^n , let us define their sum $a + b$ by

$$a + b = (a_1 + b_1, \dots, a_n + b_n).$$

One can easily verify the four conditions (A1)–(A4). This makes R^n an Abelian group. Next define a scalar multiplication in R^n by taking

$$\alpha x = (\alpha x_1, \dots, \alpha x_n)$$

for every real number α and every point $x = (x_1, \dots, x_n)$ of R^n . One can easily verify the conditions (SM1)–(SM4). Hence this scalar multiplication makes the Abelian group R^n a linear space. Its origin is the point $0 = (0, \dots, 0)$.

(2) *The real Hilbert space H .* In the example (5) of (IV, § 2), for any two points a and b of the Hilbert space H defined there, and any two real numbers α and β , the sequence $\alpha a + \beta b$ was shown to be a point of H . One can easily verify that the addition defined by $(a, b) \rightarrow a + b$ satisfies the conditions (A1)–(A4) and hence makes H an Abelian group. On the other hand, one can also verify that the assignment $(\alpha, a) \rightarrow \alpha a$ defines a scalar multiplication in H and hence makes H a linear space. Its origin is the point $0 = (0, \dots, 0, \dots)$.

(3) *The space $C(X)$.* Consider the set $C(X)$ of all bounded continuous real-valued functions defined on any given topological space X . In the example (6) in (IV, § 2), we defined a member $\alpha f + \beta g$ of $C(X)$ for any two members f, g of $C(X)$ and any two real numbers α, β . Obviously, the assignment $(f, g) \rightarrow f + g$ defines an addition in $C(X)$ and hence makes $C(X)$ an Abelian group. On the other hand, the assignment $(\alpha, f) \rightarrow \alpha f$ defines a scalar multiplication in the Abelian group $C(X)$ and hence makes $C(X)$ a linear space. Its origin is the *zero function* $0: X \rightarrow R$ defined by $0(x) = 0$ for every point x of X .

For any two nonempty subsets A, B of an arbitrary linear space X and any given real number α , we define the following subsets of X :

$$A + B = \{a + b \mid a \in A, b \in B\},$$

$$\alpha A = \{\alpha a \mid a \in A\}.$$

Thus $A + B$ and αA are nonempty subsets of the linear space X . Besides, we also mention:

$$A + \square = \square,$$

$$\alpha \square = \square,$$

$$-A = (-1)A,$$

$$A - B = A + (-1)B.$$

EXERCISES

1A. For arbitrary subsets A, B, C of a linear space X and arbitrary real numbers α, β prove the following relations:

- (a) $A + B = B + A$,
- (b) $(A + B) + C = A + (B + C)$,
- (c) $A + \{0\} = A$,
- (d) $\alpha(A + B) = \alpha A + \alpha B$,
- (e) $(\alpha + \beta)A \subset \alpha A + \beta A$,
- (f) $(\alpha\beta)A = \alpha(\beta A)$,
- (g) $1A = A$.

1B. For arbitrary subsets A, B , and C of a linear space X , prove the following statements:

- (a) $(A + B) \cap C = \square$ iff $A \cap (C - B) = \square$,
- (b) $A + B \subset C$ implies $A \subset C - B$.

2. TOPOLOGICAL LINEAR SPACES

By a *topological linear space* (over real numbers), we mean a linear space X together with a topology τ in X such that the addition

$$+ : X^2 \rightarrow X$$

and the scalar multiplication

$$\mu : R \times X \rightarrow X$$

are continuous with respect to the topology τ in X and the product topologies in X^2 and $R \times X$.

For examples, the Euclidean n -space R^n , the real Hilbert space H , and the space $C(X)$ of all bounded continuous real-valued functions on a topological space X are topological linear spaces if they are topologized by means of the metrics defined in (IV, § 2). On the other hand, the following proposition is obvious.

PROPOSITION 2.1 *The topological product of any family of topological linear spaces is a topological linear space.*

In particular, every topological power R^M of the real line R is a topological linear space. If M is not countable, then R^M is not pseudo-metrizable. Hence not every topological linear space is pseudo-metrizable.

LEMMA 2.2. *Every topological linear space is a generalized topological group.*

Proof. Let X denote any topological linear space. It follows from the continuity of the scalar multiplication μ that the function $j: X \rightarrow X$ defined by

$$j(x) = \mu(-1, x) = (-1)x = -x$$

for every $x \in X$ is continuous. Hence the function $g: X^2 \rightarrow X$ defined by

$$g(a, b) = a + j(b) = a - b$$

for every $(a, b) \in X^2$ is also continuous. According to the definition given in the example (2) of (V, § 1), X is a topological group. ||

The following corollary is a direct consequence of (2.2) and (V, 3.2).

COROLLARY 2.3. *Every topological linear space is completely regular.*

Let X denote an arbitrary topological linear space. Since X is a topological group with respect to addition, there is a natural uniformity $\mathcal{U}$ in X defined in the example (2) of (V, § 1) as follows: a subset U of X^2 is a member of $\mathcal{U}$ iff there exists a neighborhood M of the origin 0 in X such that U contains every point $(a, b) \in X^2$ satisfying

$$a - b \in M.$$

This uniformity $\mathcal{U}$ will be referred to as the *uniformity of topological linear space* X . For every local basis β of the topology τ at the origin 0 of X , the subfamily $\mathcal{B} = \{U_M \mid M \in \beta\}$ of $\mathcal{U}$ defined by

$$U_M = \{(a, b) \in X^2 \mid a - b \in M\}$$

is obviously a basis of the uniformity $\mathcal{U}$. On the other hand, since the uniformity $\mathcal{U}$ of a topological group X defines the topology τ of X , every basis $\mathcal{B}$ of $\mathcal{U}$ defines a local basis β_x at any point $x \in X$, namely

$$\beta_x = \{U(x) \mid U \in \mathcal{B}\}.$$

Hence we obtain the following proposition.

PROPOSITION 2.4. *For an arbitrary topological linear space X , the following statements are equivalent:*

- (a) *The uniformity $\mathcal{U}$ of X has a countable basis.*
- (b) *X satisfies the first axiom of countability.*
- (c) *X has a countable local basis at its origin 0 .*

The following corollary is a direct sequence of (2.4) and (V, 2.1).

COROLLARY 2.5. *A topological linear space is pseudo-metrizable iff it has a countable local basis at its origin.*

A subset K of a linear space X is said to be *convex* iff, for any two vectors a and b in K , we have

$$\lambda a + (1 - \lambda)b \in K$$

for every real number λ satisfying $0 \leq \lambda \leq 1$.

LEMMA 2.6. *If K denotes a convex subset of a linear space X containing the origin 0 of X , then*

$$\alpha K \subset \beta K$$

holds for any two nonnegative real numbers α and β satisfying $\alpha < \beta$.

Proof. Let x denote any vector in the set αK . Then, by definition, there exists a vector $a \in K$ with $\alpha a = x$. Since K is a convex set and $\lambda = \alpha/\beta$ is a real number satisfying $0 \leq \lambda \leq 1$, we have

$$\lambda a = \lambda a + (1 - \lambda)0 \in K.$$

This implies $x = \alpha a = \beta(\lambda a) \in \beta K$ and proves the lemma. ||

A topological linear space X is said to be *locally convex* iff every neighborhood of the origin 0 contains a convex neighborhood of 0 .

For examples, the Euclidean n -space R^n , the real Hilbert space H , and the space $C(X)$ are locally convex.

A subset E of a topological linear space X is said to be *bounded* iff, for every neighborhood M of the origin in X , there exists a positive integer $n \in R$ satisfying $E \subset nM$.

Before studying the problem of whether these bounded sets of X constitute a boundedness in the sense of (V, § 5), we will first consider their usefulness.

A topological linear space X is said to be *locally bounded* iff there exists a bounded neighborhood E of the origin 0 in X .

THEOREM 2.7. *Every locally bounded topological linear space has a countable local basis at its origin and hence is pseudo-metrizable.*

Proof. Let X denote an arbitrary locally bounded topological linear space. By definition, there exists a bounded neighborhood E of the origin 0 in X .

Next, for any nonzero real number α , consider the function $h_\alpha: X \rightarrow X$ defined by

$$h_\alpha(x) = \mu(\alpha, x) = \alpha x$$

for every $x \in X$. Since μ is continuous, so is h_α . Let $\beta = \alpha^{-1}$. Then the compositions $h_\alpha \circ h_\beta$ and $h_\beta \circ h_\alpha$ are the identity map on X . Hence h_α is a homeomorphism. Since $h_\alpha(0) = 0$ holds, this implies that

$$\alpha E = h_\alpha(E)$$

is a neighborhood of the origin 0 for every nonzero real number α .

It remains to prove that the sequence

$$\mathcal{F} = \{n^{-1}E \mid n \in N\}$$

of neighborhoods of the origin 0 is a local basis at 0. For this purpose, let M denote any neighborhood of 0 in X . Since E is bounded, there exists an $n \in N$ satisfying $E \subset nM$. Hence we obtain

$$n^{-1}E \subset n^{-1}(nM) = M.$$

This implies that $\mathcal{F}$ is a local basis at the origin 0. ||

Regarding the condition (B1) of a boundedness, we have the following lemma which is obvious.

LEMMA 2.8. *Every subset of any bounded set in a topological linear space is bounded.*

As to the condition (B2), we have to impose local convexity as an auxiliary condition. Precisely, we have the following lemma.

LEMMA 2.9. *In a locally convex topological linear space, the union of a finite number of bounded sets is bounded.*

Proof. Let $E_1, \dots, E_m$ denote arbitrary bounded sets in a locally convex topological linear space X . To prove that the union

$$E = E_1 \cup \dots \cup E_m$$

is bounded, let M denote any neighborhood of the origin 0 in X . Since X is locally convex, M contains a convex neighborhood K of 0. Since $E_1, \dots, E_m$ are bounded sets in X , there exist positive integers $n_1, \dots, n_m$ such that

$$E_i \subset n_i K$$

holds for all $i = 1, \dots, m$. Let

$$n = \max \{n_1, \dots, n_m\}.$$

Then it follows from (2.6) that $n_i K \subset nK$ holds for all $i = 1, \dots, m$. This implies $E \subset nK$. Since nK is contained in nM , we have

$$E \subset nK \subset nM.$$

Hence E is bounded. ||

Hence, in a locally convex topological linear space X , the bounded sets defined above constitute a boundedness $\mathcal{B}$ in X . In a general topological linear space X , the bounded sets defined above generate a boundedness $\mathcal{B}$ in X ; a subset E of X is in $\mathcal{B}$ iff E is the union of a finite number of

bounded sets defined above. In both cases, the boundedness $\mathcal{B}$ will be called the *boundedness of the topological linear space* X .

LEMMA 2.10. *The boundedness $\mathcal{B}$ of a locally bounded topological linear space X is proper and has a countable basis.*

Proof. Since X is locally bounded, there exists a bounded neighborhood E of the origin 0 in X . Let U denote the interior of E . By (2.3), X is regular and hence U contains a closed neighborhood V of 0 in X . By (2.8), both U and V are bounded neighborhoods of 0 in X .

Let B denote an arbitrary member of $\mathcal{B}$. Then B is the union of a finite number of bounded sets $B_1, \dots, B_m$ as defined above. Since V is a neighborhood of the origin 0 , there exist positive integers $n_1, \dots, n_m$ satisfying $B_i \subset n_i V$ for all $i = 1, \dots, m$. Let

$$C = \bigcup_{i=1}^m n_i V, \quad D = \bigcup_{i=1}^m n_i U.$$

Then C is a closed member of $\mathcal{B}$ and D is an open member of $\mathcal{B}$. Furthermore, we have

$$B = \bigcup_{i=1}^m B_i \subset C \subset D.$$

This proves that $\mathcal{B}$ is proper and that the subfamily $\mathcal{C}$ of $\mathcal{B}$ which consists of all unions of the finite subfamilies of the sequence

$$\{nV \mid n \in N\}$$

is a basis of $\mathcal{B}$. Since $\mathcal{C}$ is countable, this completes the proof. $\parallel$

THEOREM 2.11. (PSEUDO-METRIZATION THEOREM). *If X is a locally bounded topological linear space, then there exists a pseudo-metric*

$$d: X^2 \rightarrow R$$

which defines both the topology τ of X and the boundedness $\mathcal{B}$ of X .

Proof. In view of (2.7), (2.10), and (V, 5.11), it suffices to prove that the universe $(X, \mathcal{B})$ is locally bounded. For this purpose, let x denote an arbitrary point of X . Since the topological linear space X is locally bounded in the sense defined above, there exists a bounded open neighborhood E of the origin 0 in X . From the continuity of the scalar multiplication, it follows that the sequence $\{n^{-1}x \mid n \in N\}$ converges to the origin 0 . Since E is a neighborhood of 0 , there exists a positive integer n with $n^{-1}x \in E$. This implies that x is a point of the bounded open set nE . Hence the universe $(X, \mathcal{B})$ is locally bounded. $\parallel$

EXERCISES

- 2A. For an arbitrary topological linear space X and any nonzero real number α , prove that the assignment $x \rightarrow \alpha x$ defines a homeomorphism

$$D_\alpha: X \rightarrow X$$

of X onto itself. This homeomorphism D_α is called the *dilation* of ratio α .

- 2B. For an arbitrary topological linear space X and any given vector $a \in X$, prove that the assignment $x \rightarrow x + a$ defines a homeomorphism

$$T_a: X \rightarrow X$$

of X onto itself. This homeomorphism T_a is called the *translation* defined by a .

- 2C. Prove that a subset K of a linear space X is convex iff, for every finite number of vectors in K , say $v_0, v_1, \dots, v_n$, we have

$$\alpha_0 v_0 + \alpha_1 v_1 + \dots + \alpha_n v_n \in K$$

for arbitrary nonnegative real numbers $\alpha_0, \alpha_1, \dots, \alpha_n$ satisfying

$$\alpha_0 + \alpha_1 + \dots + \alpha_n = 1.$$

- 2D. Prove that the intersection of any family of convex sets in a linear space X is convex. For any given subset S of X , the intersection of all convex sets containing S is the smallest convex set containing S and is called the *convex hull* of S . Prove that the convex hull of S in X consists of all points of the form

$$\alpha_0 v_0 + \alpha_1 v_1 + \dots + \alpha_n v_n$$

where $v_0, v_1, \dots, v_n$ are vectors in S and $\alpha_0, \alpha_1, \dots, \alpha_n$ are nonnegative real numbers satisfying

$$\alpha_0 + \alpha_1 + \dots + \alpha_n = 1.$$

- 2E. For arbitrary convex sets A and B in a linear space X , prove that $A + B$ is convex.
- 2F. Prove that the interior $\text{Int}(K)$ and the closure $\text{Cl}(K)$ of a convex set K in a topological linear space X are convex.

3. NORMED LINEAR SPACES

By a *pseudo-norm* in a linear space X , we mean a real-valued function

$$\nu: X \rightarrow R$$

with $\nu(x)$ denoted by $\|x\|$ for every $x \in X$ and satisfying the following conditions:

(N1) *The triangle inequality:* For any two vectors a and b in X , we have

$$\|a + b\| \leq \|a\| + \|b\|.$$

(N2) For every vector x in X and any real number α , we have

$$\|\alpha x\| = |\alpha| \cdot \|x\|.$$

LEMMA 3.1. *For any pseudo-norm in a linear space X , we have*

$$(N3) \quad \|0\| = 0$$

$$(N4) \quad \|x\| \geq 0, \quad (x \in X).$$

Proof. For (N3), we have

$$\|0\| = \|00\| = |0| \cdot \|0\| = 0$$

as a consequence of (N2). To prove (N4), let x denote any vector in X . Then we have

$$0 = \|0\| = \|x + (-x)\| \leq \|x\| + \|(-1)x\| = 2\|x\|$$

as a consequence of (N1)–(N3). This implies (N4). $\square$

By a *norm* in a linear space X , we mean a pseudo-norm $\nu: X \rightarrow R$ in X which satisfies the following condition:

(N5) For any vector x in X , $\|x\| = 0$ implies $x = 0$.

Now let us consider an arbitrarily given pseudo-norm $\nu: X \rightarrow R$ in a linear space X . Define a function

$$d: X^2 \rightarrow R$$

by taking

$$d(a, b) = \|a - b\|$$

for every point (a, b) of X^2 .

LEMMA 3.2. *This function $d: X^2 \rightarrow R$ is a pseudo-metric in X .*

Proof. It suffices to verify the two conditions (M1) and (M2) in (IV, § 1). To verify (M1), let a, b, c denote arbitrary vectors in X .

Then we have

$$\begin{aligned} d(a, b) &= \|a - b\| = \|(a - c) + (-1)(b - c)\| \\ &\leq \|a - c\| + \|(-1)(b - c)\| = \|a - c\| + \|b - c\| \\ &= d(a, c) + d(b, c). \end{aligned}$$

To verify (M2), let x denote an arbitrary vector in X . Then we have

$$d(x, x) = \|x - x\| = \|0\| = 0.$$

Hence d is a pseudo-metric in X . $\parallel$

This pseudo-metric $d: X^2 \rightarrow R$ in X is called the *pseudo-metric defined by the pseudo-norm* $\nu: X \rightarrow R$. Furthermore, the topology τ in X defined by this pseudo-metric d will be referred to as the *topology defined by the pseudo-norm* $\nu: X \rightarrow R$.

Since (N5) for a pseudo-norm ν is obviously equivalent to (M5) for the pseudo-metric d defined by ν , we have the following lemma.

LEMMA 3.3. *A pseudo-norm $\nu: X \rightarrow R$ in a linear space X is a norm iff the pseudo-metric $d: X^2 \rightarrow R$ defined by ν is a metric.*

By a *pseudo-normed linear space*, we mean a linear space X together with a given pseudo-norm $\nu: X \rightarrow R$ and the topology τ in X defined by ν . In the case where the pseudo-norm ν is a norm, it is called a *normed linear space*.

The following proposition is a direct consequence of (3.3) and (IV, 2.1).

PROPOSITION 3.4. *A pseudo-normed linear space is a Fréchet space iff it is a normed linear space.*

By a *Banach space*, we mean a normed linear space X which is complete with respect to the metric $d: X^2 \rightarrow R$ defined by its norm.

EXAMPLES:

(1) *The Euclidean n -space R^n .* Define a function $\nu: R^n \rightarrow R$ by taking

$$\nu(x) = \|x\| = \sqrt{\sum_{i=1}^n x_i^2}$$

for every point $x = (x_1, \dots, x_n)$ in R^n . The condition (N1) is a straightforward consequence of the Schwarz inequality, and the condition (N2) is obvious from the definition. Besides, we clearly also have (N5). Hence ν is a norm in R^n . This norm ν will be called the *Euclidean norm* in R^n and defines the Euclidean metric d . Since R^n is complete with respect to d , it is a Banach space.

(2) *The real Hilbert space H .* Define a function $\nu: H \rightarrow R$ by taking

$$\nu(x) = \|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}$$

for every point $x = (x_1, x_2, \dots)$ in H . The condition (N1) is a consequence of the Schwarz inequality, and the condition (N2) is obvious from the definition. Besides, we clearly also have (N5). Hence ν is a norm in H . This norm ν defines the metric d in the example (5) of (IV, § 2). Since H is complete with respect to d , it is a Banach space.

(3) *The space $C(X)$.* Define a function $\nu: C(X) \rightarrow R$ by taking

$$\nu(f) = \|f\| = \sup \{|f(x)| \mid x \in X\}$$

for every member $f: X \rightarrow R$ of the set $C(X)$. One can easily verify (N1) as in the example (6) of (IV, § 2). Since (N2) and (N5) are clearly satisfied, ν is a norm in $C(X)$ and defines the metric d in the example (6) of (IV, § 2). Since $C(X)$ is complete with respect to d , it is a Banach space.

(4) *The space $\mathcal{L}_p$.* For any positive integer p , let us consider the set $\mathcal{L}_p$ of all functions $f: I \rightarrow R$ on the closed unit interval I such that the Lebesgue integral

$$\int_0^1 |f(t)|^p dt$$

exists. With addition and scalar multiplication defined in the obvious way, one can easily verify that $\mathcal{L}_p$ is a linear space. Define a function $\nu: \mathcal{L}_p \rightarrow R$ by taking

$$\nu(f) = \|f\| = \left[\int_0^1 |f(t)|^p dt \right]^{1/p}$$

for every member $f: I \rightarrow R$ of $\mathcal{L}_p$. Then (N1) follows from the second Minkowski inequality [Be, p. 215], and (N2) is obvious from the definition. Hence $\mathcal{L}_p$ is a pseudo-normed linear space. Since $\|f\| = 0$ holds for every function $f: I \rightarrow R$ which vanishes except on a subset of Lebesgue measure zero, $\mathcal{L}_p$ is not a normed linear space.

THEOREM 3.5. *Every pseudo-normed linear space is a topological linear space, both locally convex and locally bounded.*

Proof. Let us consider an arbitrary pseudo-normed linear space X with $d: X^2 \rightarrow R$ denoting the pseudo-metric defined by its pseudo-norm.

To prove the continuity of the addition, let a, b denote arbitrary vectors in X and ε be any given positive real number. Then we have

$$\begin{aligned} d(x + y, a + b) &= \|(x + y) - (a + b)\| = \|(x - a) + (y - b)\| \\ &\leq \|x - a\| + \|y - b\| = d(x, a) + d(y, b) < \varepsilon \end{aligned}$$

for all points $(x, y) \in X^2$ satisfying $d(x, a) < \varepsilon/2$ and $d(y, b) < \varepsilon/2$. Hence the addition is continuous.

To prove the continuity of the scalar multiplication, let $a \in X$, $\alpha \in R$, and a positive real number ε be arbitrarily given. Choose a positive real number γ satisfying

$$\gamma \|a\| < \frac{\varepsilon}{2},$$

and then choose a positive real number δ satisfying

$$(|\alpha| + \gamma)\delta < \frac{\varepsilon}{2}.$$

Then we have

$$\begin{aligned} d(\xi x, \alpha a) &= \|\xi x - \alpha a\| = \|\xi(x - a) + (\xi - \alpha)a\| \\ &\leq |\xi| \cdot \|x - a\| + |\xi - \alpha| \cdot \|a\| \\ &\leq (|\alpha| + \gamma)\delta + \gamma\|a\| < \varepsilon \end{aligned}$$

for all $\xi \in R$ and $x \in X$ satisfying $|\xi - \alpha| < \gamma$ and $d(x, a) < \delta$. Hence the scalar multiplication is continuous.

Thus we have proved that X is a topological linear space. It remains to prove that X is both locally convex and locally bounded.

To prove that X is locally convex, it suffices to verify that, for every positive real number r , the open neighborhood

$$B_r = \{x \in X \mid \|x\| < r\}$$

of the origin 0 is convex. For this purpose, let a and b denote arbitrary vectors in B_r . Then we have $\|a\| < r$, $\|b\| < r$, and consequently

$$\begin{aligned} \|ta + (1 - t)b\| &\leq t\|a\| + (1 - t)\|b\| \\ &< tr + (1 - t)r = r \end{aligned}$$

holds for all real numbers $t \in I$. This implies that the vector $ta + (1 - t)b$ is in B_r for all $t \in I$. Hence B_r is convex.

To prove that X is locally bounded, it suffices to verify that the open neighborhood B_1 of the origin 0 is bounded. For this purpose, let U denote an arbitrary neighborhood of the origin 0. Since the topology of X is defined by its pseudo-norm, there exists a positive integer n satisfying

$$B_{1/n} \subset U.$$

Because of the condition (N2), this implies

$$B_1 = nB_{1/n} \subset nU.$$

Since U is an arbitrary neighborhood of the origin 0, this proves that B_1 is bounded. $\parallel$

To establish the converse of (3.5), let us first define the concept of normability.

A topological linear space X is said to be *pseudo-normable* iff there exists a pseudo-norm $\nu: X \rightarrow R$ which defines the topology τ of X . In case the pseudo-norm ν is a norm, we say that the topological linear space X is *normable*.

THEOREM 3.6. *A topological linear space is pseudo-normable iff it is locally convex and locally bounded.*

Proof. Since the necessity of the condition is a direct consequence, it remains to prove the sufficiency.

Let X denote any topological linear space which is locally convex and locally bounded. Since X is locally bounded, there exists a bounded neighborhood W of the origin 0 in X . Since X is also locally convex, W contains a convex neighborhood V of 0. Since the interior of a convex set is convex according to exercise 2F, we may assume that V is open. Clearly, $-V$ is also a convex open neighborhood of 0. Hence

$$U = (-V) \cap V$$

is an open neighborhood of 0 in X which is symmetric, bounded, and convex.

Now let us define a function $\nu: X \rightarrow R$ by taking

$$\nu(x) = \|x\| = \inf \{ \lambda \geq 0 \mid x \in \lambda U \}$$

for every vector x in X . To show that ν is a pseudo-norm in X , we have to verify the condition (N1) and (N2).

To verify (N1), consider arbitrary vectors a and b in X . Let ε denote any positive real number. Since U is convex, it follows from (2.6) that we have

$$a \in (\|a\| + \varepsilon)U, \quad b \in (\|b\| + \varepsilon)U.$$

This implies that the vectors

$$a^* = (\|a\| + \varepsilon)^{-1}a, \quad b^* = (\|b\| + \varepsilon)^{-1}b$$

are in U . Since U is convex, it contains the point $ta^* + (1 - t)b^*$ for every $t \in I$. In particular, we may take

$$t = (\|a\| + \varepsilon)(\|a\| + \|b\| + 2\varepsilon)^{-1}.$$

For this particular value of t , we obtain

$$ta^* + (1 - t)b^* = (\|a\| + \|b\| + 2\varepsilon)^{-1}(a + b).$$

Since this vector is in U , we obtain

$$a + b \in (\|a\| + \|b\| + 2\varepsilon)U.$$

By the definition, this implies

$$\|a + b\| \leq \|a\| + \|b\| + 2\varepsilon.$$

Since this holds for every positive real number ε , we must have

$$\|a + b\| \leq \|a\| + \|b\|.$$

This proves (N1).

To verify (N2), consider an arbitrary vector a in X and an arbitrary real number α . Let ε denote any positive real number. Then we have

$$a \in (\|a\| + \varepsilon)U.$$

Since U is symmetric, this implies

$$\alpha a \in |\alpha|(\|a\| + \varepsilon)U.$$

By the definition, this implies

$$\|\alpha a\| \leq |\alpha|(\|a\| + \varepsilon).$$

Since this is true for every positive real number ε , we must have

$$\|\alpha a\| \leq |\alpha| \cdot \|a\|.$$

On the other hand, assume $\|a\| > 0$ and $\|a\| - \varepsilon \geq 0$. It follows from the definition of $\|a\|$ that

$$a \notin (\|a\| - \varepsilon)U$$

holds. Since U is symmetric, this implies

$$\alpha a \notin |\alpha|(\|a\| - \varepsilon)U.$$

Then it follows from the definition of $\|\alpha a\|$ that we have

$$\|\alpha a\| \geq |\alpha|(\|a\| - \varepsilon).$$

Since this is true for all positive real numbers $\varepsilon \leq \|a\|$, we must have

$$\|\alpha a\| \geq |\alpha| \cdot \|a\|.$$

Thus we have verified (N2).

Therefore, the function $\nu: X \rightarrow R$ is a pseudo-norm in the given topological linear space. It remains to prove that ν defines the topology

τ of X . For this purpose, let τ^* denote the topology in X defined by the pseudo-norm ν . We will prove $\tau^* = \tau$.

To prove $\tau \subset \tau^*$, let G denote any member of τ . To prove $G \in \tau^*$, let x be any vector in G and we will show that G contains a neighborhood of x in the topology τ^* . Since the translation T_x is a homeomorphism with respect to each of the two topologies τ and τ^* as proved in exercise 2B, we may assume without loss of generality that x is the origin 0. Since U is bounded and G is a neighborhood of 0 in the topology τ , there exists a positive integer n satisfying $U \subset nG$ and hence $n^{-1}U \subset G$. By the definition of the pseudo-norm ν , the open neighborhood

$$B_n = \{x \in X \mid \|x\| < n^{-1}\}$$

of the origin 0 in the topology τ^* is contained in $n^{-1}U$. Hence we obtain $B_n \subset G$. This proves $\tau \subset \tau^*$.

To prove $\tau^* \subset \tau$, let H denote any member of τ^* . To prove $H \in \tau$, let x be any vector in H and we will show that H contains a neighborhood of x in the topology τ . As above, we may assume without loss of generality that x is the origin 0. Because of $H \in \tau^*$, there exists a positive real number δ such that H contains the set

$$M = \{x \in X \mid \|x\| < \delta\}.$$

Choose a positive real number $\lambda < \delta$. Then it follows from the definition of ν that

$$\lambda U \subset M \subset H$$

holds. Since λU is an open neighborhood of the origin 0 in the topology τ , this proves $H \in \tau$ and hence $\tau^* \subset \tau$. ||

The following corollary is a direct consequence of (3.4) and (3.6).

COROLLARY 3.7. (NORMABILITY THEOREM.) *A topological linear space X is normable iff the following three conditions are satisfied:*

- (a) X is a Fréchet space.
- (b) X is locally convex.
- (c) X is locally bounded.

EXERCISES

- 3A. Two pseudo-norms $\nu, \nu': X \rightarrow R$ in the same linear space X are said to be *equivalent* iff they define the same topology in X . Prove that any two pseudo-norms $\nu, \nu': X \rightarrow R$ are equivalent iff, for any

given positive real number ε , there exists a positive real number δ satisfying the following two conditions:

- (a) For every $x \in X$, $\nu(x) < \delta$ implies $\nu'(x) < \varepsilon$.
- (b) For every $x \in X$, $\nu'(x) < \delta$ implies $\nu(x) < \varepsilon$.

- 3B. Prove that the pseudo-norm $\nu: X \rightarrow R$ of any pseudo-normed linear space X is continuous.
- 3C. For an arbitrary pseudo-normed linear space X , consider the reduced metric space X^* of the pseudo-metric space X . Then the points of X^* are equivalence classes of the vectors in X . Verify that the point $x^* \in X^*$ which contains a given point $x \in X$ is the class $x + \nu^{-1}(0)$. Show that X^* is a linear space with addition and scalar multiplication defined in the obvious way. With d^* denoting the metric of X^* , define a function $\nu^*: X^* \rightarrow R$ by taking

$$\nu^*(x^*) = \|x^*\| = d^*(0^*, x^*)$$

for every $x^* \in X^*$. Verify that ν^* is a norm in X^* and defines the metric d^* . Thus X^* is a normed linear space, called the *reduced normed linear space* of X .

- 3D. Prove that, in any pseudo-normed linear space X , the pseudo-metric d defined by the pseudo-norm of X defines the boundedness $\mathcal{B}$ of the topological linear space X . Hence the pseudo-norm ν constructed in the proof of (3.6) defines not only the topology τ of X but also the boundedness $\mathcal{B}$ of X , that is, a subset B of X is a member of $\mathcal{B}$ iff $\nu(B)$ is a bounded set of real numbers.

4. LINEAR FUNCTIONALS

Because of (3.6) and (3.7), we shall be concerned hereafter only with normed linear spaces.

By a *functional* of a linear space X , we mean a function $f: X \rightarrow R$ from X into the set R of real numbers. A functional $f: X \rightarrow R$ is said to be *linear* iff

$$f(\alpha a + \beta b) = \alpha f(a) + \beta f(b)$$

holds for all $a, b \in X$ and all $\alpha, \beta \in R$.

A functional $f: X \rightarrow R$ of a normed linear space X is said to be *bounded* iff there exists a positive real number k such that

$$|f(x)| \leq k \|x\|$$

holds for every $x \in X$.

THEOREM 4.1. *For an arbitrary linear functional $f: X \rightarrow R$ of a normed linear space X , the following four statements are equivalent:*

- (a) f is uniformly continuous.
- (b) f is continuous.
- (c) For every positive real number ε , there exists a positive real number δ such that $|f(x)| < \varepsilon$ holds for every $x \in X$ satisfying $\|x\| < \delta$.
- (d) f is bounded.

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c). Since f is linear, it follows that $f(0) = 0$. Therefore, the continuity of f at the origin 0 implies (c).

(c) $\Rightarrow$ (d). Assume that f is not bounded. Then, for every positive integer n , there exists a vector $x_n \in X$ satisfying

$$|f(x_n)| > n \|x_n\|.$$

This implies $x_n \neq 0$ and hence $\|x_n\| > 0$. For each $n \in N$, let

$$y_n = (n \|x_n\|)^{-1} x_n.$$

Then we have $\|y_n\| = n^{-1}$ and

$$|f(y_n)| = (n \|x_n\|)^{-1} |f(x_n)| > 1.$$

Hence (c) is not satisfied for $\varepsilon = 1$.

(d) $\Rightarrow$ (a). Assume that f is bounded. Then there exists a positive real number k such that

$$|f(x)| \leq k \|x\|$$

holds for every $x \in X$. Let ε denote any positive real number. Let $\delta = k^{-1}\varepsilon$. Then we have

$$|f(a) - f(b)| = |f(a - b)| \leq k \|a - b\| < k k^{-1}\varepsilon = \varepsilon$$

for all $a, b \in X$ satisfying $d(a, b) = \|a - b\| < \delta$. Hence f is uniformly continuous. $\parallel$

The set X^* of all continuous linear functionals of a given normed linear space X obviously forms a linear space with addition and scalar multiplication defined by

$$(f + g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha[f(x)]$$

for all $f, g \in X^*$, $x \in X$, and $\alpha \in R$.

Define a function $\nu^*: X^* \rightarrow R$ by taking

$$\nu^*(f) = \|f\| = \sup \{|f(x)| / \|x\| \mid x \neq 0\}$$

which is finite because of (b) $\Rightarrow$ (d) in (4.1).

LEMMA 4.2. *This function ν^* is a norm in X^* and hence makes X^* a normed linear space.*

Proof. We have to verify the three conditions (N1), (N2), and (N5) of a norm.

To verify (N1), let f and g be any two members of X^* . Then we have

$$\begin{aligned}\|f + g\| &= \sup \{|f(x) + g(x)|/\|x\|\} \\ &\leq \sup \{|f(x)| + |g(x)|/\|x\|\} \\ &\leq \sup \{|f(x)|/\|x\|\} + \sup \{|g(x)|/\|x\|\} \\ &= \|f\| + \|g\|.\end{aligned}$$

To verify (N2), let $f \in X^*$ and $\alpha \in R$ be arbitrarily given. Then we have

$$\begin{aligned}\|\alpha f\| &= \sup \{|\alpha[f(x)]|/\|x\|\} \\ &= \sup \{|\alpha| \cdot |f(x)|/\|x\|\} \\ &= |\alpha| \sup \{|f(x)|/\|x\|\} = |\alpha| \cdot \|f\|.\end{aligned}$$

To verify (N5), let f denote any member of X^* satisfying $\|f\| = 0$. Since

$$\sup \{|f(x)|/\|x\| \mid x \neq 0\} = \|f\| = 0,$$

it follows that $f(x) = 0$ for every $x \neq 0$. Since f is linear, we also have $f(0) = 0$. This proves that f must be the constant zero functional 0^* which is the origin of X^* . This completes the proof of (4.2). $\square$

This normed linear space X^* will be called the *dual space* of the given normed linear space X . In the literature, X^* is also called the *adjoint space* or the *conjugate space* of X .

THEOREM 4.3. *The dual space X^* of any normed linear space X is a Banach space.*

Proof. It remains to prove that X^* is complete with respect to the metric d^* defined by the norm ν^* of X^* . For this purpose, consider any Cauchy sequence

$$\sigma = \{f_n \mid n \in N\}$$

in the metric space (X^*, d^*) . Then, for any positive real number ε , there exists a positive integer k such that

$$\|f_i - f_j\| = d^*(f_i, f_j) < \varepsilon$$

holds for all $i > k$ and $j > k$.

Consider an arbitrary fixed $x \in X$. Then we have

$$|f_i(x) - f_j(x)| \leq \|f_i - f_j\| \cdot \|x\| < \varepsilon \|x\|$$

for all $i > k$ and $j > k$. This implies that $\{f_n(x) \mid n \in N\}$ is a Cauchy sequence of real numbers. Since the real line is complete, this Cauchy sequence converges to a real number

$$f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

The assignment $x \rightarrow f(x)$ for each $x \in X$ defines a functional $f: X \rightarrow R$.

To prove that f is linear, let $a, b \in X$ and $\alpha, \beta \in R$ be arbitrarily given. Then we have

$$\begin{aligned} f(\alpha a + \beta b) &= \lim_{n \rightarrow \infty} f_n(\alpha a + \beta b) \\ &= \lim_{n \rightarrow \infty} [\alpha f_n(a) + \beta f_n(b)] = \alpha f(a) + \beta f(b). \end{aligned}$$

Hence f is a linear functional.

To prove that f is continuous, it suffices to verify that f is bounded because of (4.1). Since σ is a Cauchy sequence, there exists a positive integer k such that $\|f_i - f_j\| < 1$ for all $i > k$ and $j > k$. Fix an $i > k$. Then we have

$$\|f_j\| < \|f_i\| + 1$$

for all $j > k$. Hence, for an arbitrarily given $x \in X$, we have

$$|f_j(x)| \leq (\|f_i\| + 1) \|x\|.$$

Since this is true for all $j > k$, we obtain

$$|f(x)| = \lim_{j \rightarrow \infty} |f_j(x)| \leq (\|f_i\| + 1) \|x\|$$

This proves that f is bounded and hence continuous.

Thus we have proved that f is a member of X^* . It remains to verify that the given sequence σ converges to f in X^* . For this purpose, let ε denote an arbitrarily given positive real number. Since σ is a Cauchy sequence, there is a positive integer k such that

$$\|f_i - f_j\| < \frac{\varepsilon}{3}$$

holds for all $i > k$ and $j > k$. Fix an $i > k$. By the definition of ν^* , there exists a vector $a \in X$ satisfying

$$\begin{aligned} \|f_i - f\| &< \|a\|^{-1} |f_i(a) - f(a)| + \varepsilon/3 \\ &= |f_i(\|a\|^{-1}a) - f(\|a\|^{-1}a)| + \varepsilon/3. \end{aligned}$$

Since $f(\|a\|^{-1}a)$ is defined by

$$f(\|a\|^{-1}a) = \lim_{j \rightarrow \infty} f_j(\|a\|^{-1}a),$$

there exists an integer $l > k$ such that

$$|f_j(\|a\|^{-1}a) - f(\|a\|^{-1}a)| < \varepsilon/3$$

holds for all $j > l$. Choose any $j > l$. Then we obtain

$$\begin{aligned} \|f_i - f\| &< |f_i(\|a\|^{-1}a) - f_j(\|a\|^{-1}a)| \\ &+ |f_j(\|a\|^{-1}a) - f(\|a\|^{-1}a)| + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since this is true for all $i > k$ while an arbitrary ε has been given, it follows that the given sequence σ converges to f . $\square$

EXAMPLES OF LINEAR FUNCTIONALS:

(1) Let $a = (a_1, \dots, a_n)$ denote any given point of the Euclidean n -space R^n . Define a functional $f: R^n \rightarrow R$ by taking

$$f(x) = \sum_{i=1}^n a_i x_i$$

for every point $x = (x_1, \dots, x_n)$ of R^n . To show that f is linear, let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in R^n and $\alpha, \beta \in R$ be arbitrarily given. Then we have

$$\begin{aligned} f(\alpha x + \beta y) &= \sum_{i=1}^n a_i(\alpha x_i + \beta y_i) \\ &= \alpha \sum_{i=1}^n a_i x_i + \beta \sum_{i=1}^n a_i y_i = \alpha f(x) + \beta f(y). \end{aligned}$$

To show that f is continuous, we apply the Schwarz inequality and obtain

$$|f(x)| = \left| \sum_{i=1}^n a_i x_i \right| \leq \|a\| \cdot \|x\|$$

for every $x \in R^n$. This proves that f is bounded and hence continuous. Furthermore, this implies

$$\|f\| = \sup \{|f(x)|/\|x\| \mid x \neq 0\} \leq \|a\|.$$

On the other hand, we have

$$\|f\| \geq |f(a)|/\|a\| = \|a\|^2/\|a\| = \|a\|$$

in case $a \neq 0$. Hence we have $\|f\| = \|a\|$ if $a \neq 0$. Finally, if $a = 0$, then f is the zero functional on R^n .

(2) Consider the space $C(I)$ of all (bounded) continuous functions $\phi: I \rightarrow R$. Let $\xi \in C(I)$ be arbitrarily given. Define a functional $f: C(I) \rightarrow R$ by taking

$$f(\phi) = \int_0^1 \phi(t) \xi(t) dt$$

for every $\phi \in C(I)$. To show that f is linear, let $\phi, \psi \in C(I)$ and $\alpha, \beta \in R$ be arbitrarily given; then we have

$$\begin{aligned} f(\alpha\phi + \beta\psi) &= \int_0^1 [\alpha\phi(t) + \beta\psi(t)]\xi(t) dt \\ &= \alpha \int_0^1 \phi(t)\xi(t) dt + \beta \int_0^1 \psi(t)\xi(t) dt \\ &= \alpha f(\phi) + \beta f(\psi). \end{aligned}$$

To show that f is continuous, we have

$$|f(\phi)| = \left| \int_0^1 \phi(t)\xi(t) dt \right| \leq \int_0^1 |\phi(t)\xi(t)| dt \leq \|\phi\| \int_0^1 |\xi(t)| dt$$

for every $\phi \in C(I)$. This proves that f is bounded and hence continuous. Besides, this also implies

$$\|f\| = \sup \{ |f(\phi)| / \|\phi\| \mid \phi \neq 0 \} \leq \int_0^1 |\xi(t)| dt.$$

It is possible to prove

$$\|f\| = \int_0^1 |\xi(t)| dt.$$

EXERCISES

- 4A. Let $a = (a_1, a_2, \dots)$ denote any given point of the real Hilbert space H . Prove that we may define a functional $f: H \rightarrow R$ by taking

$$f(x) = \sum_{n=1}^{\infty} a_n x_n$$

for every point $x = (x_1, x_2, \dots)$ of H . Show that f is linear and continuous and with $\|f\| = \|a\|$.

- 4B. A subset E of a linear space X is said to be a *linear subspace* of X iff $\alpha a + \beta b \in E$ holds for all $a, b \in E$ and all $\alpha, \beta \in R$. Let $f: X \rightarrow R$ denote any linear functional. Prove that

$$K = f^{-1}(0) = \{x \in X \mid f(x) = 0\}$$

is a linear subspace of X . Let a denote a given point in $X \setminus K$. Prove that, for any $x \in X$, there exist a $p \in K$ and a $\lambda \in R$ satisfying $x = \lambda a + p$.

- 4C. Let $f: E \rightarrow R$ denote a continuous linear functional on a linear subspace E of a normed linear space X . Prove the following

Hahn-Banach Theorem: There exists a continuous linear functional $g: X \rightarrow R$ satisfying $f = g \mid E$ and $\|f\| = \|g\|$.

- 4D. Let E denote a closed linear subspace of a normed linear space X and let $a \in X \setminus E$. Prove that there exists a continuous linear functional $f: X \rightarrow R$ satisfying $f(E) = 0$ and $f(a) = 1$.
- 4E. Let v denote an arbitrary nonzero vector in a normed linear space X . Prove that there exists a continuous linear functional $f: X \rightarrow R$ satisfying $f(v) = \|v\|$ and $\|f\| = 1$.

5. LINEAR OPERATORS

By a *linear operator* from a linear space X into a linear space Y , we mean a function

$$\Omega: X \rightarrow Y$$

such that

$$\Omega(\alpha a + \beta b) = \alpha \Omega(a) + \beta \Omega(b)$$

holds for all $a, b \in X$ and all $\alpha, \beta \in R$. In particular, if Y is the real line R , then Ω is a linear functional.

A linear operator $\Omega: X \rightarrow Y$ from a normed linear space X into a normed linear space Y is said to be *bounded* iff there exists a positive real number k such that

$$\|\Omega(x)\| \leq k \|x\|$$

holds for every $x \in X$.

With some obvious modifications of the proof of (4.1), one can easily prove the following theorem.

THEOREM 5.1. *For an arbitrary linear operator $\Omega: X \rightarrow Y$ from a normed linear space X into a normed linear space Y , the following four statements are equivalent:*

- (a) Ω is uniformly continuous.
- (b) Ω is continuous.
- (c) For every positive real number ε , there exists a positive real number δ such that $\|\Omega(x)\| < \varepsilon$ holds for every $x \in X$ satisfying $\|x\| < \delta$.
- (d) Ω is bounded.

For any two normed linear spaces X and Y , the set

$$\mathcal{L}(X, Y)$$

of all continuous linear operators from X into Y obviously forms a linear space with addition and scalar multiplication defined by

$$(\Omega + \Omega')(x) = \Omega(x) + \Omega'(x),$$

$$(\alpha\Omega)(x) = \alpha[\Omega(x)]$$

for all $\Omega, \Omega' \in \mathcal{L}(X, Y)$ and all $\alpha \in R$.

Define a function $\nu: \mathcal{L}(X, Y) \rightarrow R$ by taking

$$\nu(\Omega) = \|\Omega\| = \sup \{ \|\Omega(x)\| / \|x\| \mid x \neq 0 \}$$

which is finite because of (b) $\Rightarrow$ (d) in (5.1).

With some obvious modifications of the proof of (4.2), one can easily prove the following lemma.

LEMMA 5.2. *The function ν is a norm in $\mathcal{L}(X, Y)$ and hence makes $\mathcal{L}(X, Y)$ a normed linear space.*

This normed linear space $\mathcal{L}(X, Y)$ will be called the *normed linear space of all linear operators* from X into Y . In particular, in case Y is the real line, then $\mathcal{L}(X, Y)$ reduces to the dual space X^* of the normed linear space X .

With some obvious modifications of the proof of (4.3), one can easily prove the following theorem.

THEOREM 5.3. *If X is a normed linear space and Y is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space.*

As an illustrative example of linear operators, let us consider an arbitrary continuous function $K: I^2 \rightarrow R$ from the unit square $I^2 = I \times I$ into R . Consider the normed linear space $C(I)$ of all continuous functions $\phi: I \rightarrow R$. For each $\phi \in C(I)$, define a function $\Omega(\phi): I \rightarrow R$ by taking

$$[\Omega(\phi)](x) = \int_0^1 K(x, t)\phi(t) dt$$

for every $x \in I$. By means of the continuity of K and ϕ , one can easily verify that $\Omega(\phi)$ is continuous and hence is a member of $C(I)$. Thus, the assignment $\phi \rightarrow \Omega(\phi)$ defines a function

$$\Omega: C(I) \rightarrow C(I)$$

of the normed linear space $C(I)$ into itself.

To show that Ω is a linear operator, let $\phi, \psi \in C(I)$ and $\alpha, \beta \in R$ be arbitrarily given. Then we have

$$\begin{aligned} [\Omega(\alpha\phi + \beta\psi)](x) &= \int_0^1 K(x, t)[\alpha\phi(t) + \beta\psi(t)] dt \\ &= \alpha \int_0^1 K(x, t)\phi(t) dt + \beta \int_0^1 K(x, t)\psi(t) dt \\ &= \alpha[\Omega(\phi)(x)] + \beta[\Omega(\psi)(x)] \end{aligned}$$

for all $x \in I$. This implies

$$\Omega(\alpha\phi + \beta\psi) = \alpha\Omega(\phi) + \beta\Omega(\psi)$$

and hence Ω is a linear operator.

To show that Ω is continuous, we have

$$\begin{aligned} |[\Omega(\phi)](x)| &= \left| \int_0^1 K(x, t)\phi(t) dt \right| \\ &\leq \int_0^1 |K(x, t)\phi(t)| dt \\ &\leq \|\phi\| \int_0^1 |K(x, t)| dt \end{aligned}$$

for all $\phi \in C(I)$ and all $x \in I$. Since I^2 is compact and K is continuous,

$$k = \sup \left\{ \int_0^1 |K(x, t)| dt \mid x \in I \right\}$$

is finite. This implies

$$\|\Omega(\phi)\| \leq k \|\phi\|$$

for every $\phi \in C(I)$. This proves that Ω is bounded and hence continuous. Thus Ω is a continuous linear operator with $\|\Omega\| \leq k$. It is possible to prove that $\|\Omega\| = k$.

As another illustrative example of linear operators, let us consider an arbitrarily given normed linear space X . By (4.3), the dual space X^* of X is a Banach space. Then the dual space

$$X^{**} = (X^*)^*$$

of X^* is also a Banach space and will be referred to as the *bidual space* of the given normed linear space X . In the literature, X^{**} is also called the *second conjugate space* or the *second adjoint space* of the given normed linear space X .

Define a function $\Omega: X \rightarrow X^{**}$ by taking

$$[\Omega(x)](\phi) = \phi(x)$$

for every $x \in X$ and every $\phi \in X^*$.

To verify that Ω is a linear operator, let $a, b \in X$ and $\alpha, \beta \in R$ be arbitrarily given. Then we have

$$\begin{aligned} [\Omega(\alpha a + \beta b)](\phi) &= \phi(\alpha a + \beta b) = \alpha\phi(a) + \beta\phi(b) \\ &= [\alpha\Omega(a) + \beta\Omega(b)](\phi) \end{aligned}$$

for every $\phi \in X^*$. This implies

$$\Omega(\alpha a + \beta b) = \alpha\Omega(a) + \beta\Omega(b).$$

Hence Ω is a linear operator.

To verify that Ω is continuous, we have

$$|[\Omega(x)](\phi)| = |\phi(x)| \leq \|\phi\| \cdot \|x\|$$

for every $x \in X$ and every $\phi \in X^*$. This implies

$$\|\Omega(x)\| = \sup \{ |[\Omega(x)](\phi)| / \|\phi\| \mid \phi \neq 0 \} \leq \|x\|$$

for every $x \in X$. This proves that Ω is bounded and hence continuous. Thus Ω is a continuous linear operator.

On the other hand, for an arbitrarily given $x \in X$, it follows from the exercise 4E that there exists a $\phi \in X^*$ satisfying $\phi(x) = \|x\|$ and $\|\phi\| = 1$. This implies

$$|[\Omega(x)](\phi)| = |\phi(x)| = \|x\| = \|\phi\| \cdot \|x\|.$$

Consequently, we also have

$$\|\Omega(x)\| \geq \|x\|$$

for every $x \in X$.

Combining the two inequalities, we obtain

$$\|\Omega(x)\| = \|x\|$$

for every $x \in X$. As an immediate consequence of this result, we have the following theorem.

THEOREM 5.4. *The function $\Omega: X \rightarrow X^{**}$ defined by*

$$[\Omega(x)](\phi) = \phi(x)$$

for every $x \in X$ and $\phi \in X^$ is an isometric linear operator with $\|\Omega\| = 1$.*

This isometric linear operator Ω will be called the *canonical imbedding* of the given normed linear space X into its bidual space X^{**} . A normed

linear space X is said to be *reflexive* iff its canonical imbedding into its bidual space X^{**} is surjective and hence bijective. Since X^{**} is complete, we have the following corollary.

COROLLARY 5.5. *Every reflexive normed linear space is complete.*

EXERCISES

- 5A. For an arbitrary continuous linear operator $\Omega: X \rightarrow Y$ from a normed linear space X into a normed linear space Y , prove

$$\|\Omega\| = \sup \{ \|\Omega(x)\| \mid \|x\| = 1 \}.$$

- 5B. Prove that the composition $\Omega_2 \circ \Omega_1$ of any two continuous linear operators Ω_1 and Ω_2 of normed linear spaces is a continuous linear operator with

$$\|\Omega_2 \circ \Omega_1\| \leq \|\Omega_2\| \cdot \|\Omega_1\|.$$

- 5C. Prove that the Euclidean n -space R^n is a reflexive normed linear space.
- 5D. Prove that the Banach space $C(I)$ of all continuous functions on the closed unit interval I is not a reflexive normed linear space.
- 5E. Let $\Omega: X \rightarrow Y$ denote an arbitrary continuous linear operator from a normed linear space X into a normed linear space Y . Consider the dual spaces X^* and Y^* of X and Y respectively. Define a function

$$\Omega^*: Y^* \rightarrow X^*$$

by taking

$$[\Omega^*(f^*)](x) = f^*[\Omega(x)] \in R$$

for every $f^* \in Y^*$ and every $x \in X$. Prove that this function Ω^* is a continuous linear operator with

$$\|\Omega^*\| = \|\Omega\|.$$

This linear operator Ω^* is called the *dual operator* or the *adjoint operator* of Ω .

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